PART ONE

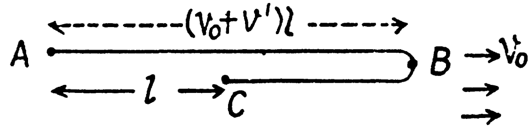
PHYSICAL FUNDAMENTALS OF MECHANICS

1.1 KINEMATICS

1.1 Let v_0 be the stream velocity and v' the velocity of motorboat with respect to water. The motorboat reached point B while going downstream with velocity $(v_0 + v')$ and then returned with velocity $(v' - v_0)$ and passed the raft at point C. Let t be the time for the raft (which flows with stream with velocity v_0) to move from point A to C, during which the motorboat moves from A to B and then from B to C.

Therefore

$$\frac{l}{v_0} = \tau + \frac{(v_0 + v')\tau - l}{(v' - v_0)}$$



On solving we get $v_0 = \dfrac{l}{2\tau}$

1.2 Let s be the total distance traversed by the point and t_1 the time taken to cover half the distance. Further let $2t$ be the time to cover the rest half of the distance.

Therefore $$\frac{s}{2} = v_0 t_1 \quad \text{or} \quad t_1 = \frac{s}{2 v_0} \tag{1}$$

and $$\frac{s}{2} = (v_1 + v_2)\, t \quad \text{or} \quad 2t = \frac{s}{v_1 + v_2} \tag{2}$$

Hence the sought average velocity

$$\langle v \rangle = \frac{s}{t_1 + 2t} = \frac{s}{[s/2 v_0] + [s/(v_1 + v_2)]} = \frac{2 v_0 (v_1 + v_2)}{v_1 + v_2 + 2 v_0}$$

1.3 As the car starts from rest and finally comes to a stop, and the rate of acceleration and deceleration are equal, the distances as well as the times taken are same in these phases of motion.

Let Δt be the time for which the car moves uniformly. Then the acceleration / deceleration time is $\dfrac{\tau - \Delta t}{2}$ each. So,

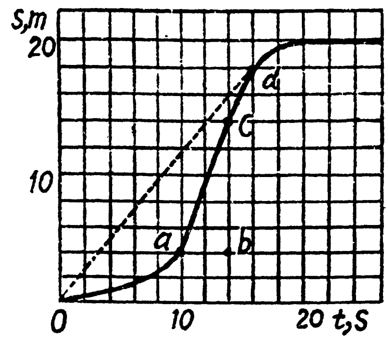
$$<v>\tau = 2\left\{\frac{1}{2}w\frac{(\tau-\Delta t)^2}{4}\right\} + w\frac{(\tau-\Delta t)}{2}\Delta t$$

or
$$\Delta t^2 = \tau^2 - \frac{4<v>\tau}{w}$$

Hence
$$\Delta t = \tau\sqrt{1-\frac{4<v>}{w\tau}} = 15 \text{ s.}$$

1.4 (a) Sought average velocity

$$<v> = \frac{s}{t} = \frac{200 \text{ cm}}{20\text{s}} = 10 \text{ cm/s}$$



(b) For the maximum velocity, $\frac{ds}{dt}$ should be maximum. From the figure $\frac{ds}{dt}$ is maximum for all points on the line ac, thus the sought maximum velocity becomes average velocity for the line ac and is equal to :

$$\frac{bc}{ab} = \frac{100 \text{ cm}}{4\text{s}} = 25 \text{ cm/s}$$

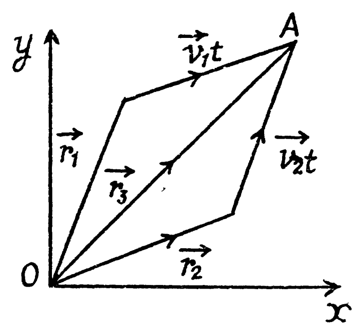
(c) Time t_0 should be such that corresponding to it the slope $\frac{ds}{dt}$ should pass through the point O (origin), to satisfy the relationship $\frac{ds}{dt} = \frac{s}{t_0}$. From figure the tangent at point d passes through the origin and thus corresponding time $t = t_0 = 16$ s.

1.5 Let the particles collide at the point A (Fig.), whose position vector is $\vec{r}_3$ (say). If t be the time taken by each particle to reach at point A, from triangle law of vector addition :

$$\vec{r}_3 = \vec{r}_1 + \vec{v}_1 t = \vec{r}_2 + \vec{v}_2 t$$

so, $$\vec{r}_1 - \vec{r}_2 = (\vec{v}_2 - \vec{v}_1)\,t \qquad (1)$$

therefore, $$t = \frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|} \qquad (2)$$



From Eqs. (1) and (2)

$$\vec{r}_1 = \vec{r}_2 - (\vec{v}_2 - \vec{v}_1)\frac{|\vec{r}_1 - \vec{r}_2|}{|\vec{v}_2 - \vec{v}_1|}$$

or, $$\frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{\vec{v}_2 - \vec{v}_1}{|\vec{v}_2 - \vec{v}_1|},$$ which is the sought relationship.

1.6 We have

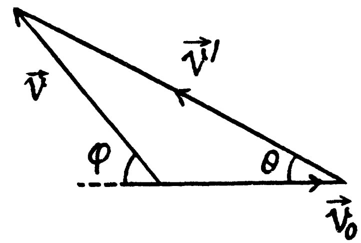
$$\vec{v}' = \vec{v} - \vec{v}_0 \qquad (1)$$

From the vector diagram [of Eq. (1)] and using properties of triangle

$$v' = \sqrt{v_0^2 + v^2 + 2\, v_0\, v \cos \varphi} = 39.7 \text{ km/hr} \qquad (2)$$

and $\dfrac{v'}{\sin(\pi - \varphi)} = \dfrac{v}{\sin \theta}$ or, $\sin \theta = \dfrac{v \sin \varphi}{v'}$

or $$\theta = \sin^{-1}\left(\frac{v \sin \varphi}{v'}\right)$$



Using (2) and putting the values of v and d

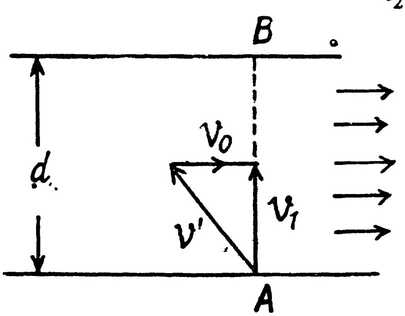
$$\theta = 19.1°$$

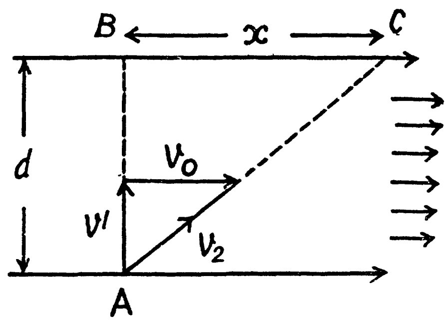
1.7 Let one of the swimmer (say 1) cross the river along AB, which is obviously the shortest path. Time taken to cross the river by the swimmer 1.

$$t_1 = \frac{d}{\sqrt{v'^2 - v_0^2}}, \text{ (where } AB = d \text{ is the width of the river)} \qquad (1)$$

For the other swimmer (say 2), which follows the quickest path, the time taken to cross the river.

$$t_2 = \frac{d}{v'} \qquad (2)$$





In the time t_2, drifting of the swimmer 2, becomes

$$x = v_0 t_2 = \frac{v_0}{v'} d, \text{ (using Eq. 2)} \qquad (3)$$

If t_3 be the time for swimmer 2 to walk the distance x to come from C to B (Fig.), then

$$t_3 = \frac{x}{u} = \frac{v_0 d}{v' u} \quad \text{(using Eq. 3)} \qquad (4)$$

According to the problem $t_1 = t_2 + t_3$

or, $$\frac{d}{\sqrt{v'^2 - v_0^2}} = \frac{d}{v'} + \frac{v_0 d}{v' u}$$

On solving we get

$$u = \frac{v_0}{\left(\dfrac{1 - v_0^2}{v'^2}\right)^{-\frac{1}{2}} - 1} = 3 \text{ km/hr.}$$

1.8 Let l be the distance covered by the boat A along the river as well as by the boat B acro the river. Let v_0 be the stream velocity and v' the velocity of each boat with respect water. Therefore time taken by the boat A in its journey

$$t_A = \frac{l}{v' + v_0} + \frac{l}{v' - v_0}$$

and for the boat B
$$t_B = \frac{l}{\sqrt{v'^2 - v_0^2}} + \frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{2l}{\sqrt{v'^2 - v_0^2}}$$

Hence,
$$\frac{t_A}{t_B} = \frac{v'}{\sqrt{v'^2 - v_0^2}} = \frac{\eta}{\sqrt{\eta^2 - 1}} \quad \left(\text{where } \eta = \frac{v'}{v}\right)$$

On substitution $t_A/t_B = 1{\cdot}8$

1.9 Let v_0 be the stream velocity and v' the velocity of boat with respect to water. A $\frac{v_0}{v'} = \eta = 2 > 0$, some drifting of boat is inevitable.

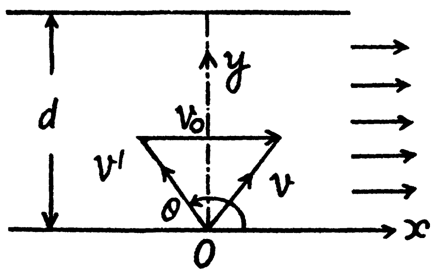
Let $\vec{v}'$ make an angle θ with flow direction. (Fig.), then the time taken to cross the rive

$$t = \frac{d}{v' \sin\theta} \quad \text{(where } d \text{ is the width of the river)}$$

In this time interval, the drifting of the boat

$$x = (v' \cos\theta + v_0)\, t$$

$$= (v' \cos\theta + v_0) \frac{d}{v' \sin\theta} = (\cot\theta + \eta \operatorname{cosec}\theta)\, d$$



For x_{min} (minimum drifting)

$$\frac{d}{d\theta}(\cot\theta + \eta \operatorname{cosec}\theta) = 0, \text{ which yields}$$

$$\cos\theta = -\frac{1}{\eta} = -\frac{1}{2}$$

Hence, $\theta = 120°$

1.10 The solution of this problem becomes simple in the frame attached with one of the bodies. Let the body thrown straight up be 1 and the other body be 2, then for the body 1 in the frame of 2 from the kinematic equation for constant acceleration :

$$\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\, t + \frac{1}{2} \vec{w}_{12} t^2$$

So, $\vec{r}_{12} = \vec{v}_{0(12)}\, t$, (because $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

or, $|\vec{r}_{12}| = |\vec{v}_{0(12)}|\, t$ (1)

But $|\vec{v}_{01}| = |\vec{v}_{02}| = v_0$

So, from properties of triangle

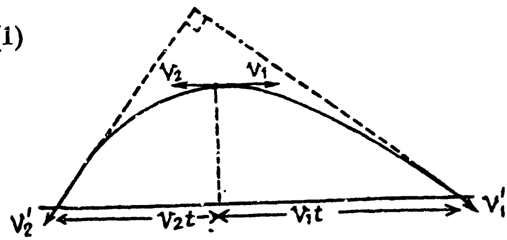
$$v_{0(12)} = \sqrt{v_0^2 + v_0^2 - 2\, v_0\, v_0 \cos(\pi/2 - \theta_0)}$$

Hence, the sought distance

$$|\vec{r}_{12}| = v_0 \sqrt{2\,(1 - \sin\theta)}\; t = 22 \text{ m}.$$

1.11 Let the velocities of the paricles (say $\vec{v}_1{}'$ and $\vec{v}_2{}'$) becomes mutually perpendicular after time t. Then their velocitis become

$$\vec{v}_1{}' = \vec{v}_1 + \vec{g}\,t\,;\ \vec{v}_2{}' = \vec{v}_2 + \vec{g}\,t \qquad (1)$$



As $\vec{v}_1{}' \perp \vec{v}_2{}'$ so, $\vec{v}_1{}' \cdot \vec{v}_2{}' = 0$

or, $\quad (\vec{v}_1 + \vec{g}\,t) \cdot (\vec{v}_2 + \vec{g}\,t) = 0$

or $\quad -v_1 v_2 + g^2 t^2 = 0$

Hence, $\quad t = \dfrac{\sqrt{v_1 v_2}}{g} \qquad (3)$

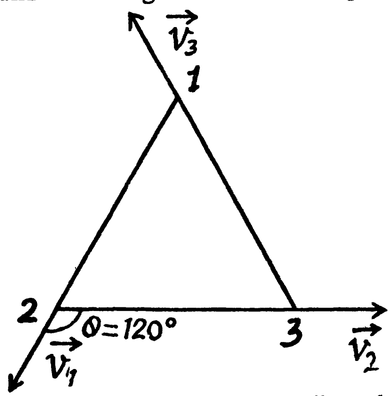
Now form the Eq. $\vec{r}_{12} = \vec{r}_{0(12)} + \vec{v}_{0(12)}\,t + \dfrac{1}{2}\vec{w}_{12}\,t^2$

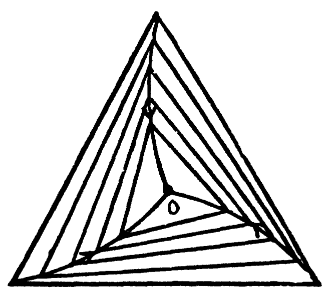
$|\vec{r}_{12}| = |\vec{v}_{0(12)}|\,t$, (because here $\vec{w}_{12} = 0$ and $\vec{r}_{0(12)} = 0$)

Hence the sought distance

$$|\vec{r}_{12}| = \frac{v_1 + v_2}{g}\sqrt{v_1 v_2} \quad (\text{as } |\vec{v}_{0(12)}| = v_1 + v_2)$$

1.12 From the symmetry of the problem all the three points are always located at the vertices of equilateral triangles of varying side length and finally meet at the centriod of the initial equilateral triangle whose side length is a, in the sought time interval (say t).





Let us consider an arbitrary equilateral triangle of edge length l (say).

Then the rate by which 1 approaches 2, 2 approches 3, and 3 approches 1, becomes :

$$\frac{-dl}{dt} = v - v\cos\left(\frac{2\pi}{3}\right)$$

On integrating :

$$-\int_a^0 dl = \frac{3v}{2}\int_0^t dt$$

$$a = \frac{3}{2}vt \quad \text{so} \quad t = \frac{2a}{3v}$$

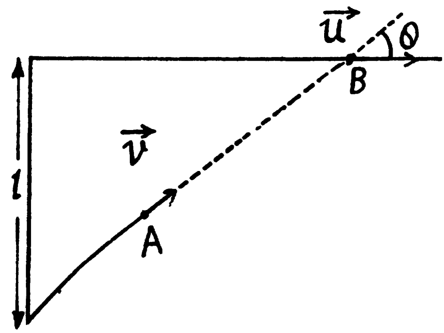
1.13 Let us locate the points A and B at an arbitrary instant of time (Fig.).
If A and B are separated by the distance s at this moment, then the points converge or point A approaches B with velocity $\frac{-ds}{dt} = v - u\cos\alpha$ where angle α varies with time.

On intergating,

$$-\int_l^0 ds = \int_0^T (v - u\cos\alpha)\,dt,$$

(where T is the sought time.)

or $$l = \int_0^T (v - u\cos\alpha)\,dt \qquad (1)$$



As both A and B cover the same distance in x-direction during the sought time interval, so the other condition which is required, can be obtained by the equation

$$\Delta x = \int v_x\,dt$$

So, $$uT = \int_0^T v\cos\alpha\,dt \qquad (2)$$

Solving (1) and (2), we get $T = \frac{ul}{v^2 - u^2}$

One can see that if $u = v$, or $u < v$, point A cannot catch B.

1.14 In the reference frame fixed to the train, the distance between the two events is obviously equal to l. Suppose the train starts moving at time $t = 0$ in the positive x direction and take the origin ($x = 0$) at the head-light of the train at $t = 0$. Then the coordinate of first event in the earth's frame is

$$x_1 = \frac{1}{2}wt^2$$

and similarly the coordinate of the second event is

$$x_2 = \frac{1}{2}w(t+\tau)^2 - l$$

The distance between the two events is obviously.

$$x_1 - x_2 = l - w\tau\,(t + \tau/2) = 0{\cdot}242 \text{ km}$$

in the reference frame fixed on the earth..

For the two events to occur at the same point in the reference frame K, moving with constant velocity V relative to the earth, the distance travelled by the frame in the time interval T must be equal to the above distance.

Thus $$V\tau = l - w\tau(t + \tau/2)$$

So, $$V = \frac{l}{\tau} - w\,(t + \tau/2) = 4{\cdot}03 \text{ m/s}$$

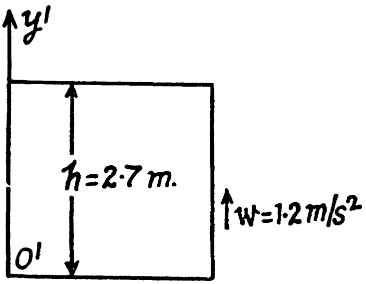
The frame K must clearly be moving in a direction opposite to the train so that if (for example) the origin of the frame coincides with the point x_1 on the earth at time t, it coincides with the point x_2 at time $t + \tau$.

1.15 (a) One good way to solve the problem is to work in the elevator's frame having the observer at its bottom (Fig.).

Let us denote the separation between floor and celing by $h = 2{\cdot}7$ m. and the acceleration of the elevator by $w = 1{\cdot}2\ \text{m/s}^2$

From the kinematical formula

$$y = y_0 + v_{0y}\, t + \frac{1}{2} w_y\, t^2 \qquad (1)$$



Here $\quad y = 0,\ y_0 = +h,\ v_{0y} = 0$

and $\quad w_y = w_{\text{bolt}\,(y)} - w_{\text{ele}\,(y)}$

$$= (-g) - (w) = -(g + w)$$

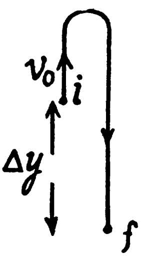
So, $\quad 0 = h + \frac{1}{2}\left\{-(g + w)\right\} t^2$

or, $\quad t = \sqrt{\frac{2h}{g + w}} = 0{\cdot}7$ s.

(b) At the moment the bolt loses contact with the elevator, it has already aquired the velocity equal to elevator, given by :

$$v_0 = (1{\cdot}2)\,(2) = 2{\cdot}4\ \text{m/s}$$

In the reference frame attached with the elevator shaft (ground) and pointing the y-axis upward, we have for the displacement of the bolt,



$$\Delta y = v_{0y}\, t + \frac{1}{2} w_y\, t^2$$

$$= v_0\, t + \frac{1}{2}(-g)\, t^2$$

or, $\quad \Delta y = (2{\cdot}4)\,(0{\cdot}7) + \frac{1}{2}(-9{\cdot}8)\,(0{\cdot}7)^2 = -0{\cdot}7$ m.

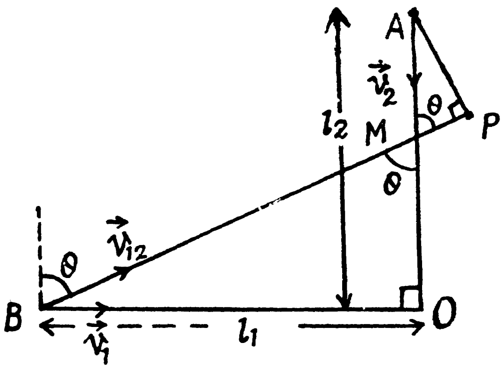
Hence the bolt comes down or displaces downward relative to the point, when it loses contact with the elevator by the amount 0·7 m (Fig.).

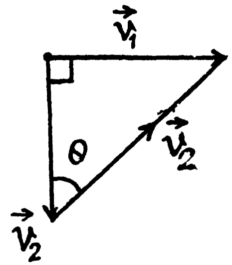
Obviously the total distance covered by the bolt during its free fall time

$$s = |\Delta y| + 2\left(\frac{v_0^2}{2g}\right) = 0{\cdot}7\ \text{m} + \frac{(2{\cdot}4)^2}{(9{\cdot}8)}\ \text{m} = 1{\cdot}3\ \text{m}.$$

1.16 Let the particle 1 and 2 be at points B and A at $t = 0$ at the distances l_1 and l_2 from intersection point O.

Let us fix the inertial frame with the particle 2. Now the particle 1 moves in relative to this reference frame with a relative velocity $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$ and its trajectory is the straight line BP. Obviously, the minimum distance between the particles is equal to the length of the perpendicular AP dropped from point A on to the straight line BP (Fig.).





From Fig. (b), $v_{12} = \sqrt{v_1^2 + v_2^2}$, and $\tan\theta = \frac{v_1}{v_2}$ (1)

The shortest distance

$$AP = AM \sin\theta = (OA - OM) \sin\theta = (l_2 - l_1 \cot\theta) \sin\theta$$

or $$AP = \left(l_2 - l_1 \frac{v_2}{v_1}\right) \frac{v_1}{\sqrt{v_1^2 + v_2^2}} = \frac{v_1 l_2 - v_2 l_1}{\sqrt{v_1^2 + v_2^2}} \quad \text{(using 1)}$$

The sought time can be obtained directly from the condition that $(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2$ is minimum. This gives $t = \frac{l_1 v_1 + l_2 v_2}{v_1^2 + v_2^2}$.

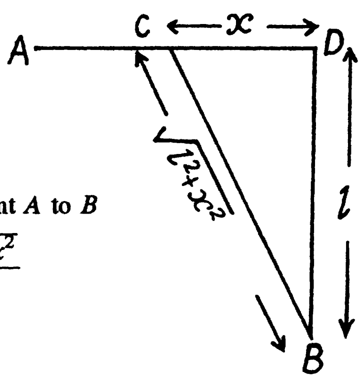
1.17 Let the car turn off the highway at a distance x from the point D.

So, $CD = x$, and if the speed of the car in the field is v, then the time taken by the car to cover the distance $AC = AD - x$ on the highway

$$t_1 = \frac{AD - x}{\eta v} \qquad (1)$$

and the time taken to travel the distance CB in the field

$$t_2 = \frac{\sqrt{l^2 + x^2}}{v} \qquad (2)$$



So, the total time elapsed to move the car from point A to B

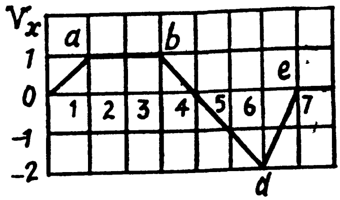
$$t = t_1 + t_2 = \frac{AD - x}{\eta v} + \frac{\sqrt{l^2 + x^2}}{v}$$

For t to be minimum

$$\frac{dt}{dx} = 0 \quad \text{or} \quad \frac{1}{v}\left[-\frac{1}{\eta} + \frac{x}{\sqrt{l^2 + x^2}}\right] = 0$$

or $$\eta^2 x^2 = l^2 + x^2 \quad \text{or} \quad x = \frac{l}{\sqrt{\eta^2 - 1}}$$

1.18 To plot $x(t)$, $s(t)$ and $w_x(t)$ let us partion the given plot $v_x(t)$ into five segments (for detailed analysis) as shown in the figure.



For the part oa : $w_x = 1$ and $v_x = t = v$

Thus, $\Delta x_1(t) = \int v_x\,dt = \int_0^t dt = \frac{t^2}{2} = s_1(t)$

Putting $t = 1$, we get, $\Delta x_1 = s = \frac{1}{2}$ unit

For the part ab :

$w_x = O$ and $v_x = v = \text{constant} = 1$

Thus $\Delta x_2(t) = \int v_x\,dt = \int_1^t dt = (t-1) = s_2(t)$

Putting $t = 3,\ \Delta x_2 = s_2 = 2$ unit

For the part $b4$: $w_x = 1$ and $v_x = 1-(t-3) = 4-t) = v$

Thus $\Delta x_3(t) = \int_3^t (4-t)\,dt = 4t - \frac{t^2}{2} - \frac{15}{2} = s_3(t)$

Putting $t = 4,\ \Delta x_3 = x_3 = \frac{1}{2}$ unit

For the part $4d$: $v_x = -1$ and $v_x = -(1-4) = 4-1$

So, $v = |v_x| = t-4$ for $t > 4$

Thus $\Delta x_4(t) = \int_4^t (1-t)\,dt = 4t - \frac{t^2}{2} - 8$

Putting $t = 6,\ \Delta x_4 = -1$

Similarly $s_4(t) = \int |v_x|\,dt = \int_4^t (t-4)\,dt = \frac{t^2}{2} - 4t + 8$

Putting $t = 6,\ s_4 = 2$ unit

For the part $d\,7$: $w_x = 2$ and $v_x = -2+2(t-6) = 2(t-7)$

$v = |v_x| = 2(7-t)$ for $t \leftarrow 7$

Now, $\Delta x(t) = \int_t^6 2\ (t-7)\,dt = t^2 - 14t + 48$

Putting $t = 4,\ \Delta x_5 = -1$

Similarly $s_5(t) = \int_t^6 2(7-t)\,dt = 14t - t^2 - 48$

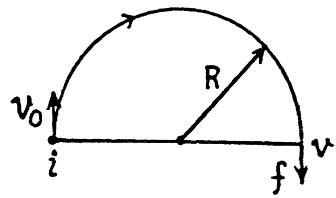
Putting $t = 7,\ s_5 = 1$

On the basis of these obtained expressions $w_x(t)$, $x(t)$ and $s(t)$ plots can be easily plotted as shown in the figure of answersheet.

1.19 (a) Mean velocity

$$<v> = \frac{\text{Total distance covered}}{\text{Time elapsed}}$$

$$= \frac{s}{t} = \frac{\pi R}{\tau} = 50 \text{ cm/s} \qquad (1)$$



(b) Modulus of mean velocity vector

$$|<\vec{v}>| = \frac{|\Delta \vec{r}|}{\Delta t} = \frac{2R}{\tau} = 32 \text{ cm/s} \qquad (2)$$

(c) Let the point moves from i to f along the half circle (Fig.) and v_0 and v be the spe at the points respectively.

We have $\qquad \frac{dv}{dt} = w_t$

or, $\qquad v = v_0 + w_t t$ (as w_t is constant, according to the problem)

So, $$<v> = \frac{\int_0^t (v_0 + w_t t)\, dt}{\int_0^t dt} = \frac{v_0 + (v_0 + w_t t)}{2} = \frac{v_0 + v}{2} \qquad ($$

So, from (1) and (3)

$$\frac{v_0 + v}{2} = \frac{\pi R}{\tau} \qquad (3$$

Now the modulus of the mean vector of total acceleration

$$|<\vec{w}>| = \frac{|\Delta \vec{v}|}{\Delta t} = \frac{|\vec{v} - \vec{v}_0|}{\tau} = \frac{v_0 + v}{\tau} \text{ (see Fig.)} \qquad (5$$

Using (4) in (5), we get :

$$|<\vec{w}>| = \frac{2\pi R}{\tau^2}$$

1.20 (a) we have $\qquad \vec{r} = \vec{a}\, t\,(1 - \alpha t)$

So, $\qquad \vec{v} = \frac{d\vec{r}}{dt} = \vec{a}(1 - 2\alpha t)$

and $\qquad \vec{w} = \frac{d\vec{v}}{dt} = -2\alpha \vec{a}$

(b) From the equation

$$\vec{r} = \vec{a}\, t\,(1 - \alpha t),$$

$$\vec{r} = 0, \text{ at } t = 0 \text{ and also at } t = \Delta t = \frac{1}{\alpha}$$

So, the sought time $\Delta t = \frac{1}{\alpha}$

As $\qquad \vec{v} = \vec{a}(1 - 2\alpha t)$

So, $$v = |\vec{v}| = \begin{cases} a\,(1 - 2\alpha t) & \text{for } t \le \frac{1}{2\alpha} \\ a\,(2\alpha t - 1) & \text{for } t > \frac{1}{2\alpha} \end{cases}$$

Hence, the sought distance

$$s = \int v\,dt = \int_0^{1/2\alpha} a\,(1 - 2\alpha t)\,dt + \int_{1/2\alpha}^{1/\alpha} a\,(2\alpha t - 1)\,dt$$

Simplifying, we get, $s = \dfrac{a}{2\alpha}$

1.21 (a) As the particle leaves the origin at $t = 0$

So, $$\Delta x = x = \int v_x\,dt \quad (1)$$

As $$\vec{v} = \vec{v_0}\left(1 - \frac{t}{\tau}\right),$$

where $\vec{v_0}$ is directed towards the +ve x-axis

So, $$v_x = v_0\left(1 - \frac{t}{\tau}\right) \quad (2)$$

From (1) and (2),

$$x = \int_0^t v_0\left(1 - \frac{t}{\tau}\right)dt = v_0 t\left(1 - \frac{t}{2\tau}\right) \quad (3)$$

Hence x coordinate of the particle at $t = 6$ s.

$$x = 10 \times 6\left(1 - \frac{6}{2 \times 5}\right) = 24\text{ cm} = 0{\cdot}24\text{ m}$$

Similarly at $t = 10$ s

$$x = 10 \times 10\left(1 - \frac{10}{2 \times 5}\right) = 0$$

and at $t = 20$ s

$$x = 10 \times 20\left(1 - \frac{20}{2 \times 5}\right) = -200\text{ cm} = -2\text{ m}$$

(b) At the moments the particle is at a distance of 10 cm from the origin, $x = \pm 10$ cm.

Putting $x = +10$ in Eq. (3)

$$10 = 10t\left(1 - \frac{t}{10}\right) \text{ or, } t^2 - 10t + 10 = 0,$$

So, $$t = t = \frac{10 \pm \sqrt{100 - 40}}{2} = 5 \pm \sqrt{15}\text{ s}$$

Now putting $x = -10$ in Eqn (3)

$$-10 = 10\left(1 - \frac{t}{10}\right),$$

On solving, $t = 5 \pm \sqrt{35}$ s

As t cannot be negative, so,

$$t = (5 + \sqrt{35})\text{ s}$$

Hence the particle is at a distance of 10 cm from the origin at three moments of time :

$$t = 5 \pm \sqrt{15}\ \text{s},\ 5 + \sqrt{35}\ \text{s}$$

(c) We have $\quad \vec{v} = \vec{v_0}\left(1 - \frac{t}{\tau}\right)$

So,
$$v = |\vec{v}| = \left.\begin{matrix} v_0\left(1 - \frac{t}{\tau}\right) & \text{for } t \le \tau \\ v_0\left(\frac{t}{\tau} - 1\right) & \text{for } t > \tau \end{matrix}\right\}$$

So
$$s = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt \text{ for } t \le \tau = v_0 t\,(1 - t/2\tau)$$

and
$$s = \int_0^\tau v_0\left(1 - \frac{t}{2}\right) dt + \int_\tau^t v_0\left(\frac{t}{\tau} - 1\right) dt \text{ for } t > \tau$$

$$= v_0 \tau\,[1 + (1 - t/\tau)^2]/2 \text{ for } t > \tau \qquad \text{(A)}$$

$$s = \int_0^4 v_0\left(1 - \frac{t}{\tau}\right) dt = \int_0^4 10\left(1 - \frac{t}{5}\right) dt = 24 \text{ cm.}$$

And for $t = 8$ s

$$s = \int_0^5 10\left(1 - \frac{t}{5}\right) dt + \int_5^8 10\left(\frac{t}{5} - 1\right) dt$$

On integrating and simplifying, we get

$$s = 34 \text{ cm.}$$

On the basis of Eqs. (3) and (4), $x(t)$ and $s(t)$ plots can be drawn as shown in the answer sheet.

1.22 As particle is in unidirectional motion it is directed along the x-axis all the time. As at $t = 0$, $x = 0$

So, $\quad \Delta x = x = s$, and $\frac{dv}{dt} = w$

Therefore, $\quad v = \alpha\sqrt{x} = \alpha\sqrt{s}$

or,
$$w = \frac{dv}{dt} = \frac{\alpha}{2\sqrt{s}}\frac{ds}{dt} = \frac{\alpha}{2\sqrt{s}}$$

$$= \frac{\alpha v}{2\sqrt{s}} = \frac{\alpha\,\alpha\sqrt{s}}{2\sqrt{s}} = \frac{\alpha^2}{2} \qquad (1)$$

As, $\quad w = \frac{dv}{dt} = \frac{\alpha^2}{2}$

On integrating, $\quad \int_0^v dv = \int_0^t \frac{\alpha^2}{2} dt \quad$ or, $\quad v = \frac{\alpha^2}{2} t \qquad (2)$

(b) Let s be the time to cover first s m of the path. From the Eq.

$$s = \int v\, dt$$

$$s = \int_0^t \frac{\alpha^2}{2}\, dt = \frac{\alpha^2}{2}\frac{t^2}{2} \qquad \text{(using 2)}$$

or $$t = \frac{2}{\alpha}\sqrt{s} \qquad (3)$$

The mean velocity of particle

$$\langle v \rangle = \frac{\int v(t)\, dt}{\int dt} = \frac{\int_0^{2\sqrt{s}/\alpha} \frac{\alpha^2}{2}\, t\, dt}{2\sqrt{s}/\alpha} = \frac{\alpha\sqrt{s}}{2}$$

1.23 According to the problem

$$-\frac{v\, dv}{ds} = a\sqrt{v} \text{ (as } v \text{ decreases with time)}$$

or, $$-\int_{v_0}^{0} \sqrt{v}\, dv = a\int_0^s ds$$

On integrating we get $s = \frac{2}{3a} v_0^{3/2}$

Again according to the problem

$$-\frac{dv}{dt} = a\sqrt{v} \text{ or } -\frac{dv}{\sqrt{v}} = a\, dt$$

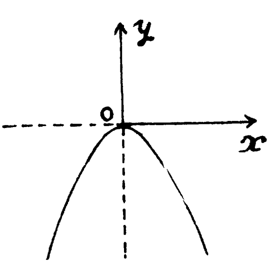
or, $$\int_{v_0}^{0} \frac{dv}{\sqrt{v}} = a\int_0^t dt$$

Thus $$t = \frac{2\sqrt{v_0}}{a}$$

1.24 (a) As $$\vec{r} = a t\vec{i} - b t^2\vec{j}$$

So, $$x = a t,\ y = -b t^2$$

and therefore $$y = \frac{-b x^2}{a^2}$$



which is Eq. of a parabola, whose graph is shown in the Fig.

(b) As $\vec{r} = a\,t\,\vec{i} - b\,t^2\,\vec{j}$

$$\vec{v} = \frac{d\vec{r}}{dt} = a\,\vec{i} - 2\,b\,t\,\vec{j} \qquad (1)$$

So, $$v = \sqrt{a^2 (-2\,b\,t)^2} = \sqrt{a^2 + 4\,b^2\,t^2}$$

Diff. Eq. (1) w.r.t. time, we get

$$\vec{w} = \frac{d\vec{v}}{dt} = -2b\,\vec{j}$$

So, $$|\vec{w}| = w = 2\,b$$

(c) $$\cos\alpha = \frac{\vec{v}\cdot\vec{w}}{v\,w} = \frac{(a\,\vec{i} - 2\,b\,t\,\vec{j})\cdot(-2\,b\,\vec{j})}{\left(\sqrt{a^2 + 4\,b^2\,t^2}\right) 2b}$$

or, $$\cos\alpha = \frac{2\,b\,t}{\sqrt{a^2 + 4\,b^2\,t^2}},$$

so, $$\tan\alpha = \frac{a}{2\,b\,t}$$

or, $\alpha = \tan^{-1}\left(\frac{a}{2\,b\,t}\right)$

(d) The mean velocity vector

$$<\vec{v}> = \frac{\int \vec{v}\,dt}{\int dt} = \frac{\int_0^t (a\,\vec{i} - 2\,b\,t\,\vec{j})\,dt}{t} = a\,\vec{i} - b\,t\,\vec{j}$$

Hence, $$|<\vec{v}>| = \sqrt{a^2 + (-b\,t)^2} = \sqrt{a^2 + b^2\,t^2}$$

1.25 (a) We have

$$x = a\,t \text{ and } y = a\,t\,(1 - \alpha\,t) \qquad (1)$$

Hence, $y(x)$ becomes,

$$y = \frac{a\,x}{a}\left(1 - \frac{\alpha\,x}{a}\right) = x - \frac{\alpha}{a}x^2 \text{ (parabola)}$$

(b) Differentiating Eq. (1) we get

$$v_x = a \text{ and } v_y = a\,(1 - 2\,\alpha\,t) \qquad (2)$$

So, $$v = \sqrt{v_x^2 + v_y^2} = a\sqrt{1 + (1 - 2\alpha t)^2}$$

Diff. Eq. (2) with respect to time

$$w_x = 0 \text{ and } w_y = -2a\alpha$$

So, $$w = \sqrt{w_x^2 + w_y^2} = 2a\alpha$$

(c) From Eqs. (2) and (3)

We have $$\vec{v} = a\vec{i} + a(1 - 2\alpha t)\vec{j} \text{ and } \vec{w} = 2a\alpha\vec{j}$$

So, $$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\vec{v}\cdot\vec{w}}{vw} = \frac{-a(1 - 2\alpha t_0)2a\alpha}{a\sqrt{1 + (1 - 2\alpha t_0)^2}\,2a\alpha}$$

On simplifying. $$1 - 2\alpha t_0 = \pm 1$$

As, $$t_0 \neq 0,\ t_0 = \frac{1}{\alpha}$$

1.26 Differentiating motion law : $x = a\sin\omega t$, $y = a(1 - \cos\omega t)$, with respect to time, $v_x = a\omega\cos\omega t$, $v_y = a\omega\sin\omega t$

So, $$\vec{v} = a\omega\cos\omega t\,\vec{i} + a\omega\sin\omega t\,\vec{j} \tag{1}$$

and $$v = a\omega = \text{Const.} \tag{2}$$

Differentiating Eq. (1) with respect to time

$$\vec{w} = \frac{d\vec{v}}{dt} = -a\omega^2\sin\omega t\,\vec{i} + a\omega^2\cos\omega t\,\vec{j} \tag{3}$$

(a) The distance s traversed by the point during the time τ is given by

$$s = \int_0^\tau v\,dt = \int_0^\tau a\omega\,dt = a\omega\tau \quad (\text{using } 2)$$

(b) Taking inner product of $\vec{v}$ and $\vec{w}$

We get, $$\vec{v}\cdot\vec{w} = (a\omega\cos\omega t\,\vec{i} + a\omega\sin\omega t\,\vec{j})\cdot(a\omega^2\sin\omega t(-i) + a\omega^2\cos\omega t - \vec{j})$$

So, $$\vec{v}\cdot\vec{w} = -a^2\omega^2\sin\omega t\cos\omega t + a^2\omega^3\sin\omega t\ \cos\omega t = 0$$

Thus, $\vec{v} \perp \vec{w}$, i.e., the angle between velocity vector and acceleration vector equals $\frac{\pi}{2}$.

1.27 Accordiing to the problem

$$\vec{w} = w(-\vec{j})$$

So, $$w_x = \frac{dv_x}{dt} = 0 \text{ and } w_y = \frac{dv_y}{dt} = -w \tag{1}$$

Differentiating Eq. of trajectory, $y = ax - bx^2$, with respect to time

$$\frac{dy}{dt} = \frac{a\,dx}{dt} - 2bx\frac{dx}{dt} \tag{2}$$

So, $$\left.\frac{dy}{dt}\right|_{x=0} = a\left.\frac{dx}{dt}\right|_{x=0}$$

Again differentiating with respect to time

$$\frac{d^2y}{dt^2} = \frac{a\,d^2x}{dt^2} - 2b\left(\frac{dx}{dt}\right)^2 - 2bx\frac{d^2x}{dt^2}$$

or, $$-w = a\,(0) - 2b\left(\frac{dx}{dt}\right)^2 - 2bx\,(0) \text{ (using 1)}$$

or, $$\frac{dx}{dt} = \sqrt{\frac{w}{2b}} \text{ (using 1)} \qquad (3$$

Using (3) in (2) $$\left.\frac{dy}{dt}\right|_{x=0} = a\sqrt{\frac{w}{2b}} \qquad (4$$

Hence, the velocity of the particle at the origin

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2_{x=0} + \left(\frac{dy}{dt}\right)^2_{x=0}} = \sqrt{\frac{w}{2b} + a^2\frac{w}{2b}} \text{ (using Eqns (3) and (4))}$$

Hence, $$v = \sqrt{\frac{w}{2b}(1+a^2)}$$

1.28 As the body is under gravity of constant accelration $\vec{g}$, it's velocity vector and displacemen vectors are:

$$\vec{v} = \vec{v}_0 + \vec{g}t \qquad (1)$$

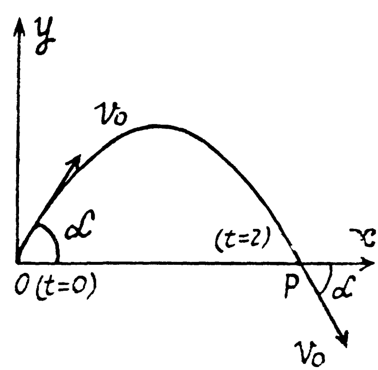
and $$\Delta\vec{r} = \vec{r} = \vec{v}_0 t + \frac{1}{2}gt^2 \quad (\vec{r} = 0 \text{ at } t = 0) \qquad (2)$$

So, $\langle\vec{v}\rangle$ over the first t seconds

$$\langle\vec{v}\rangle = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}}{t} = \vec{v}_0 + \frac{\vec{g}t}{2} \qquad (3)$$

Hence from Eq. (3), $\langle\vec{v}\rangle$ over the first t seconds

$$\langle\vec{v}\rangle = \vec{v}_0 + \frac{\vec{g}}{2}\tau \qquad (4)$$



For evaluating t, take

$$\vec{v}\cdot\vec{v} = (\vec{v}_0 + \vec{g}t)\cdot(\vec{v}_0 + \vec{g}t) = v_0^2 + 2\,(\vec{v}_0\cdot\vec{g})\,t + g^2t^2$$

or, $$v^2 = v_0^2 + (\vec{v}_0\cdot\vec{g})\,t + g^2t^2$$

But we have $v = v_0$ at $t = 0$ and

Also at $t = \tau$ (Fig.) (also from energy conservation)

Hence using this propety in Eq. (5)

$$v_0^2 = v_0^2 + 2\,(\vec{v}_0 \cdot \vec{g})\,\tau + g^2\tau^2$$

As $\tau \neq 0$, so, $\tau = -\dfrac{2\,(\vec{v}_o \cdot \vec{g})}{g^2}$

Putting this value of τ in Eq. (4), the average velocity over the time of flight

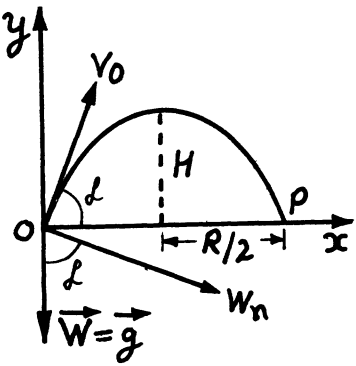
$$\langle \vec{v} \rangle = \vec{v}_0 - \vec{g}\,\frac{(\vec{v}_0 \cdot \vec{g})}{g^2}$$

1.29 The body thrown in air with velocity v_0 at an angle α from the horizontal lands at point P on the Earth's surface at same horizontal level (Fig.). The point of projection is taken as origin, so, $\Delta x = x$ and $\Delta y = y$

(a) From the Eq. $\Delta y = v_{0y} t + \frac{1}{2} w_y t^2$

$$0 = v_0 \sin\alpha\,\tau - \frac{1}{2} g\tau^2$$

As $\tau \neq 0$, so, time of motion $\tau = \dfrac{2\,v_0 \sin\alpha}{g}$



(b) At the maximum height of ascent, $v_y = 0$

so, from the Eq. $v_y^2 = v_{0y}^2 + 2\,w_y\,\Delta y$

$$0 = (v_0 \sin\alpha)^2 - 2\,g\,H$$

Hence maximum height $H = \dfrac{v_0^2 \sin^2\alpha}{2g}$

During the time of motion the net horizontal displacement or horizontal range, will be obtained by the equation

$$\Delta x = v_{0x}\,t + \frac{1}{2} w_x \tau^2$$

or, $$R = v_0 \cos\alpha\,\tau - \frac{1}{2}(0)\,\tau^2 = v_0 \cos\alpha\,\tau = \frac{v_0^2 \sin 2\alpha}{g}$$

when $R = H$

$$\frac{v_0^2 \sin^2\alpha}{g} = \frac{v_0^2 \sin^2\alpha}{2g} \quad \text{or} \quad \tan\alpha = 4, \text{ so, } \alpha = \tan^{-1} 4$$

(c) For the body, $x(t)$ and $y(t)$ are

$$x = v_0 \cos\alpha\, t \qquad (1)$$

and $$y = v_0 \sin \alpha \, t - \frac{1}{2} g t^2 \qquad (2)$$

Hence putting the value of t from (1) into (2) we get,

$$y = v_0 \sin \alpha \left(\frac{x}{v_0 \cos \alpha} \right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos \alpha} \right)^2 = x \tan \alpha - \frac{g x^2}{2 v_0^2 \cos^2 \alpha},$$

Which is the sought equation of trajectory i.e. $y(x)$

(d) As the body thrown in air follows a curve, it has some normal acceleration at all the moments of time during it's motion in air.

At the initial point $(x = 0, y = 0)$, from the equation :

$$w_n = \frac{v^2}{R}, \text{ (where } R \text{ is the radius of curvature)}$$

$$g \cos \alpha = \frac{v_0^2}{R_0} \text{ (see Fig.) or } R_0 = \frac{v_0^2}{g \cos \alpha}$$

At the peak point $v_y = 0$, $v = v_x = v_0 \cos \alpha$ and the angential acceleration is zero.

Now from the Eq. $$w_n = \frac{v^2}{R}$$

$$g = \frac{v_0^2 \cos^2 \alpha}{R}, \quad \text{or } R = \frac{v_0^2 \cos^2 \alpha}{g}$$

Note : *We may use the formula of curvature radius of a trajectory $y(x)$, to solve part (d),*

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left| d^2 y / dx^2 \right|}$$

1.30 We have, $v_x = v_0 \cos \alpha$, $v_y = v_0 \sin \alpha - gt$

As $\vec{v} \uparrow\uparrow \hat{u}_t$ all the moments of time.

Thus $$v^2 = v_t^2 - 2 g t v_0 \sin \alpha + g^2 t^2$$

Now, $$w_t = \frac{dv_t}{dt} = \frac{1}{2 v_t} \frac{d}{dt} (v_t^2) = \frac{1}{v_t} (g^2 t - g v_0 \sin \alpha)$$

$$= -\frac{g}{v_t} (v_0 \sin \alpha - g t) = -g \frac{v_y}{v_t}$$

Hence $$| w_t | = g \frac{| v_y |}{v}$$

Now $$w_n = \sqrt{w^2 - w_t^2} = \sqrt{g^2 - g^2 \frac{v_y^2}{v_t^2}}$$

or $$w_n = g \frac{v_x}{v_t} \left(\text{where } v_x = \sqrt{v_t^2 - v_y^2} \right)$$

As $\vec{v} \uparrow\uparrow \hat{v}_t$, during time of motion

$$w_v = w_t = -g\frac{v_y}{v}$$

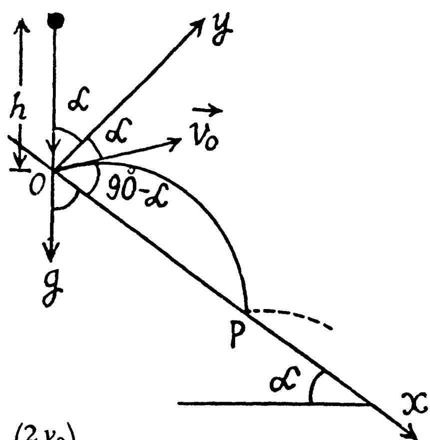
On the basis of obtained expressions or facts the sought plots can be drawn as shown in the figure of answer sheet.

1.31 The ball strikes the inclined plane (Ox) at point O (origin) with velocity $v_0 = \sqrt{2gh}$ (1)

As the ball elastically rebounds, it recalls with same velocity v_0, at the same angle α from the normal or y axis (Fig.). Let the ball strikes the incline second time at P, which is at a distance l (say) from the point O, along the incline. From the equation

$$y = v_{0y}\,t + \frac{1}{2} w_y\, t^2$$

$$0 = v_0 \cos\alpha\,\tau - \frac{1}{2} g \cos\alpha\, t^2$$



where τ is the time of motion of ball in air while moving from O to P.

As $\tau \neq 0$, so, $\tau = \dfrac{2\,v_0}{g}$ (2)

Now from the equation.

$$x = v_{0x}\,t + \frac{1}{2} w_x\, t^2$$

$$l = v_0 \sin\alpha\,\tau + \frac{1}{2} g \sin\alpha\,\tau^2$$

so, $$l = v_0 \sin\alpha\left(\frac{2\,v_0}{g}\right) + \frac{1}{2} g \sin\alpha\left(\frac{2\,v_0}{g}\right)$$

$$= \frac{4\,v_0^2 \sin\alpha}{g} \quad \text{(using 2)}$$

Hence the sought distance, $l = \dfrac{4\,(2\,gh)\sin\alpha}{g} = 8h\sin\alpha$ (Using Eq. 1)

1.32 Total time of motion

$$\tau = \frac{2\,v_0 \sin\alpha}{g} \quad \text{or} \quad \sin\alpha = \frac{\tau g}{2\,v_0} = \frac{9{\cdot}8\,\tau}{2\times 240} \qquad (1)$$

and horizontal range

$$R = v_0 \cos\alpha\,\tau \quad \text{or} \quad \cos\alpha = \frac{R}{v_0\,\tau} = \frac{5100}{240\,\tau} = \frac{85}{4\,\tau} \qquad (2)$$

From Eqs. (1) and (2)

$$\frac{(9{\cdot}8)^2\,\tau^2}{(480)^2} + \frac{(85)^2}{(4\,\tau^2)^2} = 1$$

On simplifying $\tau^4 - 2400\,\tau^2 + 1083750 = 0$

Solving for τ^2 we get :

$$\tau^2 = \frac{2400 \pm \sqrt{1425000}}{2} = \frac{2400 \pm 1194}{2}$$

Thus $\tau = 42{\cdot}39 \text{ s} = 0{\cdot}71 \text{ min}$ and

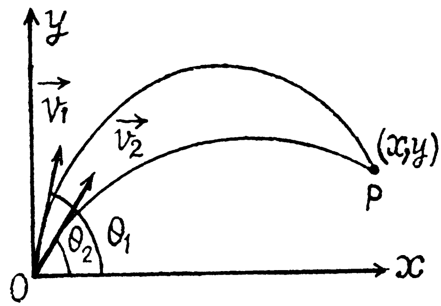
$\tau = 24{\cdot}55 \text{ s} = 0{\cdot}41 \text{ min}$ depending on the angle α.

1.33 Let the shells collide at the point $P(x, y)$. If the first shell takes t s to collide with second and Δt be the time interval between the firings, then

$$x = v_0 \cos\theta_1 \, t = v_0 \cos\theta_2 (t - \Delta t) \quad (1)$$

and $y = v_0 \sin\theta_1 \, t - \frac{1}{2} g t^2$

$$= v_0 \sin\theta_2 (t - \Delta t) - \frac{1}{2} g (t - \Delta t)^2 \quad (2)$$



From Eq. (1) $t = \dfrac{\Delta t \cos\theta_2}{\cos\theta_2 - \cos\theta_1}$ (3)

From Eqs. (2) and (3)

$$\Delta t = \frac{2 v_0 \sin(\theta_1 - \theta_2)}{g(\cos\theta_2 + \cos\theta_1)} \quad \text{as } \Delta t \neq 0$$

1.34 According to the problem

(a) $\dfrac{dy}{dt} = v_0$ or $dy = v_0 \, dt$

Integrating $\displaystyle\int_0^y dy = v_0 \int_0^t dt$ or $y = v_0 t$ (1)

And also we have $\dfrac{dx}{dt} = ay$ or $dx = a y \, dt = a v_0 t \, dt$ (using 1)

So, $\displaystyle\int_0^x dx = a v_0 \int_0^t t \, dt$, or, $x = \frac{1}{2} a v_0 t^2 = \frac{1}{2} \frac{a y^2}{v_0}$ (using 1)

(b) According to the problem

$$v_y = v_0 \text{ and } v_x = a y \quad (2)$$

So, $v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + a^2 y^2}$

Therefore $w_t = \dfrac{dv}{dt} = \dfrac{a^2 y}{\sqrt{v_0^2 + a y^2}} \dfrac{dy}{dt} = \dfrac{a^2 y}{\sqrt{1 + (ay/v_0)^2}}$

Diff. Eq. (2) with respect to time.

$$\frac{d v_y}{dt} = w_y = 0 \text{ and } \frac{d v_x}{dt} = w_x = a \frac{dy}{dt} = a v_0$$

So, $w = |w_x| = a v_0$

Hence $w_n = \sqrt{w^2 - w_t^2} = \sqrt{a^2 v_0^2 - \dfrac{a^4 y^2}{1 + (ay/v_0)^2}} = \dfrac{a v_0}{\sqrt{1 + (ay/v_0)^2}}$

1.35 (a) The velocity vector of the particle

$$\vec{v} = a\vec{i} + bx\vec{j}$$

So, $$\frac{dx}{dt} = a : \frac{dy}{dt} = bx \qquad (1)$$

From (1) $$\int_0^x dx = a\int_0^t dt \quad \text{or,} \quad x = at \qquad (2)$$

And $$dy = bx\,dt = bat\,dt$$

Integrating $$\int_0^y dy = ab\int_0^t t\,dt \quad \text{or,} \quad y = \frac{1}{2}ab\,t^2 \qquad (3)$$

From Eqs. (2) and (3), we get, $y = \dfrac{b}{2a}x^2$ (4)

(b) The curvature radius of trajectory $y(x)$ is :

$$R = \frac{\left[1 + (dy/dx)^2\right]^{\frac{3}{2}}}{\left|d^2y/dx^2\right|} \qquad (5)$$

Let us differentiate the path Eq. $y = \dfrac{b}{2a}x^2$ with respect to x,

$$\frac{dy}{dx} = \frac{b}{a}x \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{b}{a} \qquad (6)$$

From Eqs. (5) and (6), the sought curvature radius :

$$R = \frac{a}{b}\left[1 + \left(\frac{b}{a}x\right)^2\right]^{\frac{3}{2}}$$

1.36 In accordance with the problem

$$w_t = \vec{a}\cdot\vec{\tau}$$

But $$w_t = \frac{v\,dv}{ds} \quad \text{or} \quad v\,dv = w_t\,ds$$

So, $$v\,dv = (\vec{a}\cdot\vec{\tau})\,ds = \vec{a}\cdot d\vec{r}$$

or, $v\,dv = a\vec{i}\cdot d\vec{r} = a\,dx$ (because $\vec{a}$ is directed towards the x-axis)

So, $$\int_0^v v\,dv = a\int_0^x dx$$

Hence $$v^2 = 2ax \quad \text{or,} \quad v = \sqrt{2ax}$$

1.37 The velocity of the particle $v = at$

So, $$\frac{dv}{dt} = w_t = a \qquad (1)$$

And $$w_n = \frac{v^2}{R} = \frac{a^2 t^2}{R} \text{ (using } v = at) \qquad (2)$$

From $$s = \int v\, dt$$

$$2\pi R \eta = a \int_0^t v\, dt = \frac{1}{2} a t^2$$

So, $$\frac{4\pi\eta}{a} = \frac{t^2}{R} \qquad (3)$$

From Eqs. (2) and (3) $w_n = 4\pi a \eta$

Hence $w = \sqrt{w_t^2 + w_n^2}$

$$= \sqrt{a^2 + (4\pi a \eta)^2} = a\sqrt{1 + 16\pi^2 \eta^2} = 0{\cdot}8 \text{ m/s}^2$$

1.38 According to the problem

$$|w_t| = |w_n|$$

For $v(t)$, $$\frac{-dv}{dt} = \frac{v^2}{R}$$

Integrating this equation from $v_0 \le v \le v$ and $0 \le t \le t$

$$-\int_{v_0}^{v} \frac{dv}{v^2} = \frac{1}{R}\int_0^t dt \text{ or, } v = \frac{v_0}{\left(1 + \frac{v_0 t}{R}\right)}$$

Now for $v(s)$, $-\frac{v\, dv}{ds} = \frac{v^2}{R}$, Integrating this equation from $v_0 \le v \le v$ and $0 \le s \le s$

SO, $$\int_{v_0}^{v} \frac{dv}{v} = -\frac{1}{R}\int_0^s ds \text{ or, } \ln \frac{v}{v_0} = -\frac{s}{R}$$

Hence $$v = v_0 e^{-s/R} \qquad (2)$$

(b) The normal acceleration of the point

$$w_n = \frac{v^2}{R} = \frac{v^2 e^{-2s/R}}{R} \text{ (using 2)}$$

And as accordance with the problem

$$|w_t| = |w_n| \text{ and } w_t \hat{u}_t \perp w_n \hat{u}_n$$

so, $$w = \sqrt{2}\, w_n = \sqrt{2}\, \frac{v_0^2}{R} e^{-2s/R} = \sqrt{2}\, \frac{v^2}{R}$$

1.39 From the equation $v = a\sqrt{s}$

$$w_t = \frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2}, \text{ and}$$

$$w_n = \frac{v^2}{R} = \frac{a^2 s}{R}$$

As w_t is a positive constant, the speed of the particle increases with time, and the tangential acceleration vector and velocity vector coincides in direction.

Hence the angle between $\vec{v}$ and $\vec{w}$ is equal to between $w_t \hat{u}_t$ an $\vec{w}$, and α can be found by means of the formula : $\tan\alpha = \frac{|w_n|}{|w_t|} = \frac{a^2 s/R}{a^2/2} = \frac{2s}{R}$

1.40 From the equation $l = a\sin\omega t$

$$\frac{dl}{dt} = v = a\,\omega\cos\omega t$$

So, $$w_t = \frac{dv}{dt} = -a\,\omega^2\sin\omega t, \text{ and} \quad (1)$$

$$w_n = \frac{v^2}{R} = \frac{a^2\omega^2\cos^2\omega t}{R} \quad (2)$$

(a) At the point $l = 0$, $\sin\omega t = 0$ and $\cos\omega t = \pm 1$ so, $\omega t = 0, \pi$ etc.

Hence $$w = w_n = \frac{a^2\omega^2}{R}$$

Similarly at $l = \pm a$, $\sin\omega t = \pm 1$ and $\cos\omega t = 0$, so, $w_n = 0$

Hence $$w = |w_t| = a\,\omega^2$$

1.41 As $w_t = a$ and at $t = 0$, the point is at rest

So, $v(t)$ and $s(t)$ are, $v = at$ and $s = \frac{1}{2}at^2$ (1)

Let R be the curvature radius, then

$$w_n = \frac{v^2}{R} = \frac{a^2t^2}{R} = \frac{2\,as}{R} \text{ (using 1)}$$

But according to the problem

$$w_n = bt^4$$

So, $$bt^4 = \frac{a^2t^2}{R} \text{ or, } R = \frac{a^2}{bt^2} = \frac{a^2}{2\,bs} \text{ (using 1)} \quad (2)$$

Therefore $$w = \sqrt{w_t^2 + w_n^2} = \sqrt{a^2 + (2\,as/R)^2} = \sqrt{a^2 + \left(4\,bs^2/a^2\right)^2} \text{ (using 2)}$$

Hence $$w = a\sqrt{1 + \left(4\,bs^2/a^3\right)^2}$$

1.42 (a) Let us differentiate twice the path equation $y(x)$ with respect to time.

$$\frac{dy}{dt} = 2\,ax\frac{dx}{dt};\ \frac{d^2y}{dt^2} = 2a\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right]$$

Since the particle moves uniformly, its acceleration at all points of the path is normal and at the point $x = 0$ it coincides with the direction of derivative d^2y/dt^2. Keeping in mind that at the point $x = 0$, $\left|\frac{dx}{dt}\right| = v$,

We get
$$w = \left|\frac{d^2y}{dt^2}\right|_{x=0} = 2\,a\,v^2 = w_n$$

So,
$$w_n = 2\,a\,v^2 = \frac{v^2}{R},\ \text{ or }\ R = \frac{1}{2a}$$

Note that we can also calculate it from the formula of problem (1.35 b)

(b) Differentiating the equation of the trajectory with respect to time we see that

$$b^2x\,\frac{dx}{dt} + a^2y\,\frac{dy}{dt} = 0 \tag{1}$$

which implies that the vector $(b^2x\vec{i} + a^2y\vec{j})$ is normal to the velocity vector $\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$ which, of course, is along the tangent. Thus the former vactor is along the normal and the normal component of acceleration is clearly

$$w_n = \frac{b^2x\frac{d^2x}{dt^2} + a^2y\frac{d^2y}{dt^2}}{(b^4x^2 + a^4y^2)^{1/2}}$$

on using $w_n = \vec{w}\cdot\vec{n}/|\vec{n}|$. At $x = 0$, $y = \pm\,b$ and so at $x = 0$

$$w_n = \pm\left.\frac{d^2y}{dt^2}\right|_{x=0}$$

Differentiating (1)

$$b^2\left(\frac{dx}{dt}\right)^2 + b^2x\left(\frac{d^2x}{dt^2}\right) + a^2\left(\frac{dy}{dt}\right)^2 + a^2y\left(\frac{d^2y}{dt^2}\right) = 0$$

Also from (1)
$$\frac{dy}{dt} = 0 \text{ at } x = 0$$

So
$$\left(\frac{dx}{dt}\right) = \pm\,v \text{ (since tangential velocity is constant } = v\text{)}$$

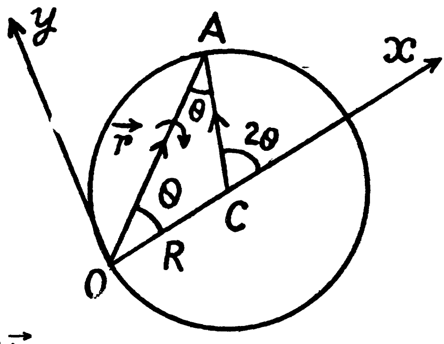
Thus
$$\left(\frac{d^2y}{dt^2}\right) = \pm\,\frac{b}{a^2}v^2$$

and
$$|w_n| = \frac{bv^2}{a^2} = \frac{v^2}{R}$$

This gives $R = a^2/b$.

1.43 Let us fix the co-ordinate system at the point O as shown in the figure, such that the radius vector $\vec{r}$ of point A makes an angle θ with x axis at the moment shown.

Note that the radius vector of the particle A rotates clockwise and we here take line ox as reference line, so in this case obviously the angular velocity $\omega = \left(-\frac{d\theta}{dt}\right)$ taking anticlockwise sense of angular displacement as positive.



Also from the geometry of the triangle OAC

$$\frac{R}{\sin\theta} = \frac{r}{\sin(\pi - 2\theta)} \text{ or, } r = 2R\cos\theta.$$

Let us write,

$$\vec{r} = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} = 2R\cos^2\theta\,\vec{i} + R\sin 2\theta\,\vec{j}$$

Differentiating with respect to time.

$$\frac{d\vec{r}}{dt} \text{ or } \vec{v} = 2R\,2\cos\theta(-\sin\theta)\frac{d\theta}{dt}\vec{i} + 2R\cos 2\theta\frac{d\theta}{dt}\vec{j}$$

$$\text{or, } \vec{v} = 2R\left(\frac{-d\theta}{dt}\right)[\sin 2\theta\,\vec{i} - \cos 2\theta\,\vec{j}]$$

$$\text{or, } \vec{v} = 2R\omega(\sin 2\theta\,\vec{i} - \cos^2\theta\,\vec{j})$$

So, $\quad |\vec{v}|$ or $v = 2\omega R = 0{\cdot}4\text{ m/s}.$

As ω is constant, v is also constant and $w_t = \frac{dv}{dt} = 0$,

So, $$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R = 0{\cdot}32\text{ m/s}^2$$

Alternate : From the Fig. the angular velocity of the point A, with respect to centre of the circle C becomes

$$-\frac{d(2\theta)}{dt} = 2\left(\frac{-d\theta}{dt}\right) = 2\omega = \text{constant}$$

Thus we have the problem of finding the velocity and acceleration of a particle moving along a circle of radius R with constant angular velocity 2ω.

Hence $\quad v = 2\omega R$ and

$$w = w_n = \frac{v^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R$$

1.44 Differentiating $\varphi(t)$ with respect to time

$$\frac{d\varphi}{dt} = \omega_z = 2at \qquad (1)$$

For fixed axis rotation, the speed of the point A:

$$v = \omega R = 2atR \text{ or } R = \frac{v}{2at} \qquad (2)$$

Differentiating with respect to time

$$w_t = \frac{dv}{dt} = 2aR = \frac{v}{t}, \text{ (using 1)}$$

But
$$w_n = \frac{v^2}{R} = \frac{v^2}{v/2at} = 2atv \quad \text{(using 2)}$$

So,
$$w = \sqrt{w_t^2 + w_n^2} = \sqrt{(v/t)^2 + (2atv)^2}$$

$$= \frac{v}{t}\sqrt{1 + 4a^2t^4}$$

1.45 The shell acquires a constant angular acceleration at the same time as it accelerates linearly. The two are related by (assuming both are constant)

$$\frac{w}{l} = \frac{\beta}{2\pi n}$$

Where w = linear acceleration and β = angular acceleration

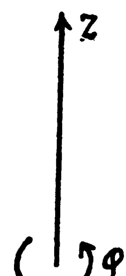
Then, $\omega = \sqrt{2 \cdot \beta\, 2\pi n} = \sqrt{2 \cdot \frac{w}{l}(2\pi n)^2}$

But $v^2 = 2wl$, hence finally

$$\omega = \frac{2\pi n v}{l}$$

1.46 Let us take the rotation axis as z-axis whose positive direction is associated with the positive direction of the cordinate φ, the rotation angle, in accordance with the right-hand screw rule (Fig.)

(a) Defferentiating $\varphi(t)$ with respect to time.



$$\frac{d\varphi}{dt} = a - 3bt^2 = \omega_z \qquad (1) \text{ and}$$

$$\frac{d^2\varphi}{dt^2} = \frac{d\omega_z}{dt} = \beta_z = -6bt \qquad (2)$$

From (1) the solid comes to stop at $\Delta t = t = \sqrt{\frac{a}{3b}}$

The angular velocity $\omega = a - 3bt^2$, for $0 \le t \le \sqrt{a/3b}$

So,
$$<\omega> = \frac{\int \omega\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} (a - 3bt^2)\, dt}{\int_0^{\sqrt{a/3b}} dt} = [at - bt^3]_0^{\sqrt{a/3b}} \Big/ \sqrt{a/3b} = 2a/3$$

Similarly $\beta = |\beta_z| = 6bt$ for all values of t.

So, $$< \beta > = \frac{\int \beta\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} 6bt\, dt}{\int_0^{\sqrt{a/3b}} dt} = \sqrt{3ab}$$

(b) From Eq. (2) $\beta_z = -6bt$

So, $$(\beta_z)_{t = \sqrt{a/3b}} = -6b\sqrt{\frac{a}{3b}} = -2\sqrt{ab}$$

Hence $$\beta = \left| (\beta_z)_{t = \sqrt{a/3b}} \right| = 2\sqrt{3ab}$$

1.47 Angle α is related with $|w_t|$ and w_n by means of the fomula :

$$\tan\alpha = \frac{w_n}{|w_t|}, \text{ where } w_n = \omega^2 R \text{ and } |w_t| = \beta R \tag{1}$$

where R is the radius of the circle which an arbitrary point of the body circumscribes.

From the given equation $\beta = \frac{d\omega}{dt} = at$ (here $\beta = \frac{d\omega}{dt}$, as β is positive for all values of t)

Integrating within the limit $\int_0^{\omega} d\omega = a\int_0^t t\, dt$ or, $\omega = \frac{1}{2}at^2$

So, $$w_n = \omega^2 R = \left(\frac{at^2}{2}\right)^2 R = \frac{a^2t^4}{4}R$$

and $$|w_t| = \beta R = atR$$

Putting the values of $|w_t|$ and w_n in Eq. (1), we get,

$$\tan\alpha = \frac{a^2t^4R/4}{atR} = \frac{at^3}{4} \text{ or, } t = \left[\left(\frac{4}{a}\right)\tan\alpha\right]^{1/3}$$

1.48 In accordance with the problem, $\beta_z < 0$

Thus $-\frac{d\omega}{dt} = k\sqrt{\omega}$, where k is proportionality constant

or, $$-\int_{\omega_0}^{\omega} \frac{d\omega}{\sqrt{\omega}} = k\int_0^t dt \text{ or, } \sqrt{\omega} = \sqrt{\omega_0} - \frac{kt}{2} \tag{1}$$

When $\omega = 0$, total time of rotation $t = \tau = \frac{2\sqrt{\omega_0}}{k}$

$$\text{Average angular velocity } < \omega > = \frac{\int \omega\, dt}{\int dt} = \frac{\int_0^{2\sqrt{\omega_0}/k} \left(\omega_0 + \frac{k^2 t^2}{4} - k t \sqrt{\omega_0} \right) dt}{2\sqrt{\omega_0}/k}$$

$$\text{Hence } < \omega > = \left[\omega_0 t + \frac{k^2 t^3}{12} - \frac{k}{2} \sqrt{\omega_0}\, t^2 \right]_0^{2\sqrt{\omega_0}/k} \Big/ 2\frac{\sqrt{\omega_0}}{k} = \omega_0/3$$

1.49 We have $\omega = \omega_0 - a\,\varphi = \frac{d\varphi}{dt}$

Integratin this Eq. within its limit for $(\varphi)\, t$

$$\int_0^{\varphi} \frac{d\varphi}{\omega_0 - k\varphi} = \int_0^{t} dt \text{ or, } \ln \frac{\omega_0 - k\varphi}{\omega_0} = -kt$$

Hence $$\varphi = \frac{\omega_0}{k}(1 - e^{-kt}) \qquad (1)$$

(b) From the Eq., $\omega = \omega_0 - k\varphi$ and Eq. (1) or by differentiating Eq. (1)

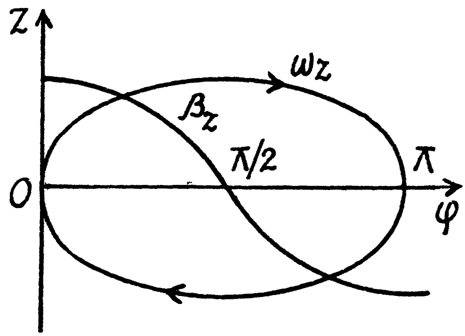
$$\omega = \omega_0 e^{-kt}$$

1.50 Let us choose the positive direction of z-axis (stationary rotation axis) along the vector $\vec{\beta}_0$. In accordance with the equation

$$\frac{d\omega_z}{dt} = \beta_z \quad \text{or} \quad \omega_z \frac{d\omega_z}{d\varphi} = \beta_z$$

or, $\omega_z\, d\omega_z = \beta_z\, d\varphi = \beta \cos\varphi\, d\varphi$,

Integrating this Eq. within its limit for $\omega_z(\varphi)$



$$\text{or,} \quad \int_0^{\omega_z} d\omega_z = \beta_0 \int_0^{\varphi} \cos\varphi\, d\varphi$$

$$\text{or,} \quad \frac{\omega_z^2}{2} = \beta_0 \sin\varphi$$

$$\text{Hence} \quad \omega_z = \pm\sqrt{2\beta_0 \sin\varphi}$$

The plot $\omega_z(\varphi)$ is shown in the Fig. It can be seen that as the angle φ grows, the vector $\vec{\omega}$ first increases, coinciding with the direction of the vector $\vec{\beta}_0$ ($\omega_z > 0$), reaches the maximum at $\varphi = \varphi/2$, then starts decreasing and finally turns into zero at $\varphi = \pi$. After that the body starts rotating in the opposite direction in a similar fashion ($\omega_z < 0$). As a result, the body will oscillate about the position $\varphi = \varphi/2$ with an amplitude equal to $\pi/2$.

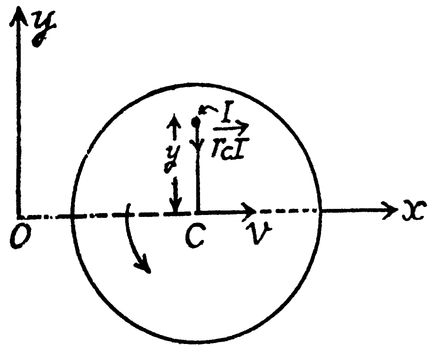
1.51 Rotating disc moves along the x-axis, in plane motion in $x-y$ plane. Plane motion of a solid can be imagined to be in pure rotation about a point (say I) at a certain instant known as instantaneous centre of rotation. The instantaneous axis whose positive sense is directed along $\vec{\omega}$ of the solid and which passes through the point I, is known as instantaneous axis of rotation.

Therefore the velocity vector of an arbitrary point (P) of the solid can be represented as :

$$\vec{v}_p = \vec{\omega} \times \vec{r}_{PI} \qquad (1)$$

On the basis of Eq. (1) for the C. M. (C) of the disc

$$\vec{v}_c = \vec{\omega} \times \vec{r}_{cI} \qquad (2)$$



According to the problem $\vec{v}_c \uparrow\uparrow \vec{i}$ and $\vec{\omega} \uparrow\uparrow \vec{k}$ i.e. $\vec{\omega} \perp x-y$ plane, so to satisy the Eqn. (2) $\vec{r}_{CI}$ is directed along $(-\vec{j})$. Hence point I is at a distance $r_{CI} = y$, above the centre of the disc along $y-$axis. Using all these facts in Eq. (2), we get

$$v_C = \omega y \text{ or } y = \frac{v_c}{\omega} \qquad (3)$$

(a) From the angular kinematical equation

$$\omega_z = \omega_{0z} + \beta_z t \qquad (4)$$

$$\omega = \beta t.$$

On the other hand $x = v t$, (where x is the x coordinate of the C.M.)

or, $$t = \frac{x}{v} \qquad (5)$$

From Eqs. (4) and (5), $\omega = \dfrac{\beta x}{v}$

Using this value of ω in Eq. (3) we get $y = \dfrac{v_c}{\omega} = \dfrac{v}{\beta x/v} = \dfrac{v^2}{\beta x}$ (hyperbola)

(b) As centre C moves with constant acceleration w, with zero initial velocity

So, $$x = \frac{1}{2} w t^2 \text{ and } v_c = w t$$

Therefore, $$v_c = w\sqrt{\frac{2x}{w}} = \sqrt{2 x w}$$

Hence $$y = \frac{v_c}{\omega} = \frac{\sqrt{2 w x}}{\omega} \text{ (parabola)}$$

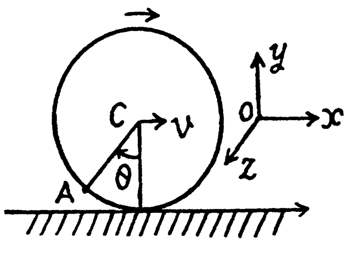
1.52 The plane motion of a solid can be imagined as the combination of translation of the C.M. and rotation about C.M.

So, we may write $\vec{v}_A = \vec{v}_C + \vec{v}_{AC}$

$= \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$ (1) and

$\vec{w}_A = \vec{w}_C + \vec{w}_{AC}$

$= \vec{w}_C + \omega^2(-\vec{r}_{AC}) + (\vec{\beta} \times \vec{r}_{AC})$ (2)



$\vec{r}_{AC}$ is the position of vector of A with respect to C.

In the problem $v_C = v =$ constant, and the rolling is without slipping i.e., $v_C = v = \omega R$,
So, $w_C = 0$ and $\beta = 0$. Using these conditions in Eq. (2)

$$\vec{w}_A = \omega^2(-\vec{r}_{AC}) = \omega^2 R(-\hat{u}_{AC}) = \frac{v^2}{R}(-\hat{u}_{AC})$$

Here, $\hat{u}_{AC}$ is the unit vector directed along $\vec{r}_{AC}$.

Hence $w_A = \frac{v^2}{R}$ and $\vec{w}_A$ is directed along $(-\hat{u}_{AC})$ or directed toward the centre of the wheel.

(b) Let the centre of the wheel move toward right (positive x-axis) then for pure tolling on the rigid horizontal surface, wheel will have to rotate in clockwise sense. If ω be the angular velocity of the wheel then $\omega = \frac{v_C}{R} = \frac{v}{R}$.

Let the point A touches the horizontal surface at $t = 0$, further let us locate the point A at $t = t$,

When it makes $\theta = \omega t$ at the centre of the wheel.

From Eqn. (1) $\vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$

$$= v\vec{i} + \omega(-\vec{k}) \times [R\cos\theta(-\vec{j}) + R\sin\theta(-\vec{i})]$$

or, $\vec{v}_A = v\vec{i} + \omega R[\cos\omega t(-\vec{i}) + \sin\omega t\vec{j}]$

$$= (v - \cos\omega t)\vec{i} + v\sin\omega t\vec{j} \quad (\text{as } v = \omega R)$$

So, $v_A = \sqrt{(v - v\cos\omega t)^2 + (v\sin\omega t)^2}$

$$= v\sqrt{2(1-\cos\omega t)} = 2v\sin(\omega t/2)$$

Hence distance covered by the point A during $T = 2\pi/\omega$

$$s = \int v_A\,dt = \int_0^{2\pi/\omega} 2v\sin(\omega t/2)\,dt = \frac{8v}{\omega} = 8R.$$

1.53 Let us fix the co-ordinate axis xyz as shown in the fig. As the ball rolls without slipping along the rigid surface so, on the basis of the solution of problem 1.52 :

Thus
$$\left.\begin{array}{l}\vec{v}_0 = \vec{v}_c + \vec{\omega} \times \vec{r}_{oc} = 0 \\ v_c = \omega R \text{ and } \vec{\omega} \uparrow\uparrow (-\vec{k}) \text{ as } \vec{v}_c \uparrow\uparrow \vec{i}\end{array}\right\} \quad (1)$$

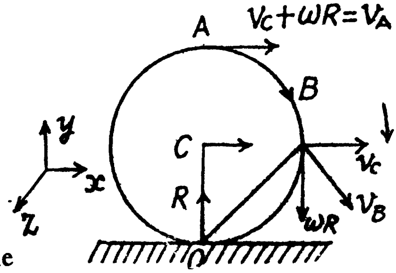
and $\quad \vec{\omega}_c + \vec{\beta} \times \vec{r}_{oc} = 0$

$w_c = \beta R$ and $\vec{\beta} \uparrow\uparrow (-\vec{k})$ as $\vec{w}_c \uparrow\uparrow \vec{i}$

At the position corresponding to that of Fig., in accordance with the problem,

$$w_c = w, \text{ so } v_c = wt$$

and $\quad \omega = \frac{v_c}{R} = \frac{wt}{R}$ and $\beta = \frac{w}{R}$ (using 1)



(a) Let us fix the co-ordinate system with the frame attached with the rigid surface as shown in the Fig.

As point O is the instantaneous centre of rotation of the ball at the moment shown in Fig.

so, $\quad \vec{v}_0 = 0,$

Now, $\quad \vec{v}_A = \vec{v}_C + \vec{\omega} \times \vec{r}_{AC}$

$$= v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{j}) = (v_C + \omega R)\vec{i}$$

So, $\quad \vec{v}_A = 2 v_C \vec{i} = 2 wt \vec{i}$ (using 1)

Similalry $\vec{v}_B = \vec{v}_C + \vec{\omega} \times \vec{r}_{BC} = v_C \vec{i} + \omega(-\vec{k}) \times R(\vec{i})$

$$= v_C \vec{i} + \omega R(-\vec{j}) = v_C \vec{i} + v_C(-\vec{j})$$

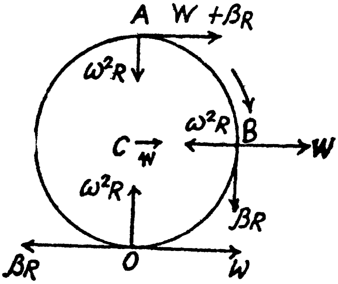
So, $v_B = \sqrt{2}\, v_c = \sqrt{2}\, wt$ and $\vec{v}_B$ is at an angle 45° from both $\vec{i}$ and $\vec{j}$ (Fig.)

(b) $\vec{w}_0 = \vec{w}_C + \omega^2(-\vec{r}_{oc}) + \vec{\beta} \times \vec{r}_{OC}$

$$= \omega^2(-\vec{r}_{OC}) = \frac{v_C^2}{R}(-\hat{u}_{OC}) \text{ (using 1)}$$

where $\hat{u}_{OC}$ is the unit vector along $\vec{r}_{OC}$

so, $\quad w_0 = \frac{v_0^2}{R} = \frac{w^2 t^2}{R}$ (using 2) and $\vec{w}_0$ is directed towards the centre of the ball



Now $\vec{w}_A = \vec{w}_C + \omega^2(-\vec{r}_{AC}) + \vec{\beta} \times \vec{r}_{AC}$

$= w\vec{i} + \omega^2 R(-\vec{j}) + \beta(-\vec{k}) \times R\vec{j}$

$= (w + \beta R)\vec{i} + \frac{v_c^2}{R}(-\vec{j})$ (using 1) $= 2w\vec{i} + \frac{w^2t^2}{R}(-\vec{j})$

So, $\quad w_A = \sqrt{4w^2 + \frac{w^4 t^4}{R^2}} = 2w\sqrt{1 + \left(\frac{wt^2}{2R}\right)^2}$

Similarly $\quad \vec{w}_B = \vec{w}_c + \omega^2(-\vec{r}_{BC}) + \vec{\beta} \times \vec{r}_{BC}$

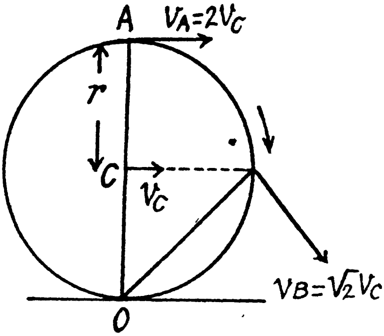
$$= w\vec{i} + \omega^2 R(-\vec{i}) + \beta(-\vec{k}) \times R(\vec{i})$$

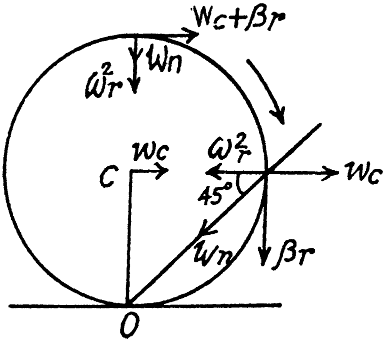
$$= \left(v - \frac{v_C^2}{R}\right)\vec{i} + \beta R(-\vec{j}) \text{ (using 1)}$$

$$= \left(w - \frac{w^2 t^2}{R}\right)\vec{i} + w(-\vec{j}) \text{ (using 2)}$$

So,
$$w_B = \sqrt{\left(w - \frac{w^2 t^2}{R}\right)^2 + w^2}$$

1.54 Let us draw the kinematical diagram of the rolling cylinder on the basis of the solutioı of problem 1.53.





As, an arbitrary point of the cylinder follows a curve, its normal acceleration and radius of curvature are related by the well known equation

$$w_n = \frac{v^2}{R}$$

so, for point A,
$$w_{A(n)} = \frac{v_A^2}{R_A}$$

or,
$$R_A = \frac{4 v_c^2}{\omega_r^2} = 4r \text{ (because } v_c = \omega r, \text{ for pure rolling)}$$

Similarly for point B,

$$w_{B(n)} = \frac{v_B^2}{R_B}$$

$$\omega^2 r \cos 45° = \frac{(\sqrt{2}\, v_c)^2}{R_B},$$

or,
$$R_B = 2\sqrt{2}\,\frac{v_C^2}{\omega^2 r} = 2\sqrt{2}\, r$$

1.55 The angular velocity is a vector as infinitesimal rotation commute. Then the relative angular velocity of the body 1 with respect to the body 2 is clearly.

$$\vec{\omega}_{12} = \vec{\omega}_1 - \vec{\omega}_2$$

as for relative linear velocity. The relative acceleration of 1 w.r.t. 2 is

$$\left(\frac{d\vec{\omega}_1}{dt}\right)_{S'}$$

where S' is a frame corotating with the second body and S is a space fixed frame with origin coinciding with the point of intersection of the two axes,

but $$\left(\frac{d\vec{\omega}_1}{dt}\right)_S = \left(\frac{d\vec{\omega}_1}{dt}\right)_{S'} + \vec{\omega}_2 \times \vec{\omega}_1$$

Since S' rotates with angular velocity $\vec{w}_2$. However $\left(\frac{d\vec{\omega}_1}{dt}\right)_S = 0$ as the first body rotates with constant angular velocity in space, thus

$$\vec{\beta}_{12} = \vec{\omega}_1 \times \vec{\omega}_2.$$

Note that for any vector $\vec{b}$, the relation in space forced frame (k) and a frame (k') rotating with angular velocity $\vec{\omega}$ is

$$\left.\frac{d\vec{b}}{dt}\right|_K = \left.\frac{d\vec{b}}{dt}\right|_{K'} + \vec{\omega} \times \vec{b}$$

1.56 We have $\vec{\omega} = at\,\vec{i} + bt^2\vec{j}$ (1)

So, $\omega = \sqrt{(at)^2 + (bt^2)^2}$, thus, $\omega|_{t=10s} = 7.81$ rad/s

Differentiating Eq. (1) with respect to time

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = a\vec{i} + 2\,bt\,\vec{j} \qquad (2)$$

So, $$\beta = \sqrt{a^2 + (2\,bt)^2}$$

and $$\beta|_{t=10s} = 1\cdot 3 \text{ rad/s}^2$$

(b) $$\cos\alpha = \frac{\vec{\omega}\cdot\vec{\beta}}{\omega\beta} = \frac{(at\,\vec{i} + bt^2\vec{j})\cdot(a\vec{i} + 2\,bt\,\vec{j})}{\sqrt{(at)^2 + (bt^2)^2}\sqrt{a^2 + (2bt)^2}}$$

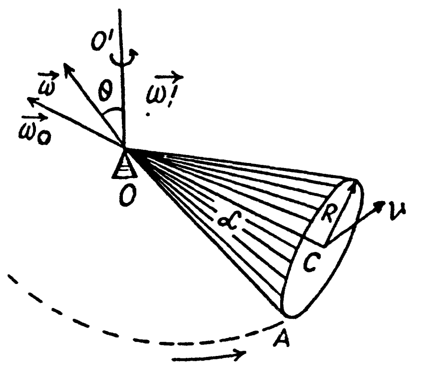
Putting the values of (a) and (b) and taking $t = 10$s, we get

$$\alpha = 17°$$

1.57 (a) Let the axis of the cone (OC) rotates in anticlockwise sense with constant angular velocity $\vec{\omega}'$ and the cone itself about it's own axis (OC) in clockwise sense with angular velocity $\vec{\omega}_0$ (Fig.). Then the resultant angular velocity of the cone.

$$\vec{\omega} = \vec{\omega}' + \vec{\omega}_0 \qquad (1)$$

As the rolling is pure the magnitudes of the vectors $\vec{\omega}'$ and $\vec{\omega}_0$ can be easily found from Fig.



$$\omega' = \frac{v}{R\cot\alpha},\quad \omega_0 = v/R \qquad (2)$$

As $\vec{\omega}' \perp \vec{\omega}_0$, from Eq. (1) and (2)

$$\omega = \sqrt{\omega'^2 + \omega_0^2}$$

$$\sqrt{\left(\frac{v}{R \cot \alpha}\right)^2 + \left(\frac{v}{R}\right)^2} = \frac{v}{R \cos \alpha} = 2{\cdot}3 \text{ rad/s}$$

(b) Vector of angular acceleration

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} \quad (\text{as } \vec{\omega}' = \text{constant.})$$

The vector $\vec{\omega}_0$ which rotates about the OO' axis with the angular velocity $\vec{\omega}'$, retains i magnitude. This increment in the time interval dt is equal to

$$|d\vec{\omega}_0| = \omega_0 \cdot \omega' \, dt \text{ or in vector form } d\vec{\omega}_0 = (\vec{\omega}' \times \vec{\omega}_0)\, dt.$$

Thus $\qquad \vec{\beta} = \vec{\omega}' \times \vec{\omega}_0$ (3

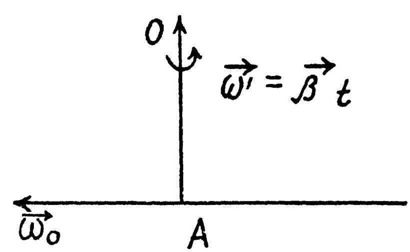
The magnitude of the vector $\vec{\beta}$ is equal to

$$\beta = \omega' \omega_0 \, (\text{as } \vec{\omega}' \perp \vec{\omega}_0)$$

So, $\qquad \beta = \dfrac{v}{R \cot \alpha} \dfrac{v}{R} = \dfrac{v^2}{R^2} \tan \alpha = 2{\cdot}3 \text{ rad/s}$

1.58 The axis AB acquired the angular velocity

$$\vec{\omega}' = \vec{\beta}_o t \qquad (1)$$



Using the facts of the solution of 1.57, the angular velocity of the body

$$\omega = \sqrt{\omega_o^2 + \omega'^2}$$

$$= \sqrt{\omega_0^2 + \beta_0^2 t^2} = 0{\cdot}6 \text{ rad/s}$$

And the angular acceleration.

$$\vec{\beta} = \frac{d\vec{\omega}}{dt} = \frac{d(\vec{\omega}' + \vec{\omega}_0)}{dt} = \frac{d\vec{\omega}'}{dt} + \frac{d\vec{\omega}_0}{dt}$$

But $\qquad \dfrac{d\vec{\omega}_0}{dt} = \vec{\omega}' \times \vec{\omega}_0, \text{ and } \dfrac{d\vec{\omega}'}{dt} = \vec{\beta}_0 t$

So, $\qquad \vec{\beta} = (\vec{\beta}_0 t \times \vec{\omega}_0) + \vec{\beta}_0$

As, $\vec{\beta}_0 \perp \vec{\omega}_0$ so, $\beta = \sqrt{(\omega_0 \beta_0 t)^2 + \beta_0^2} = \beta_0 \sqrt{1 + (\omega_0 t)^2} = 0{\cdot}2 \text{ rad/s}^2$

1.2 THE FUNDAMENTAL EQUATION OF DYNAMICS

1.59 Let R be the constant upward thurst on the aerostat of mass m, coming down with a constant acceleration w. Applying Newton's second law of motion for the aerostat in projection form

$$F_y = mw_y$$

$$mg - R = mw \tag{1}$$

Now, if Δm be the mass, to be dumped, then using the Eq. $F_y = mw_y$

$$R - (m - \Delta m)\, g = (m - \Delta m)\, w, \tag{2}$$

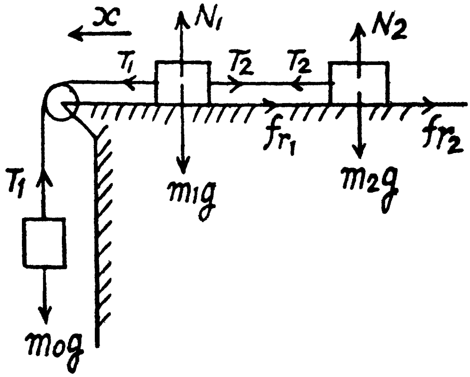
From Eqs. (1) and (2), we get, $\quad \Delta m = \dfrac{2\, mw}{g + w}$

1.60 Let us write the fundamental equation of dynamics for all the three blocks in terms of projections, having taken the positive direction of x and y axes as shown in Fig; and using the fact that kinematical relation between the accelerations is such that the blocks move with same value of acceleration (say w)

$$m_0\, g - T_1 = m_0\, w \tag{1}$$

$$T_1 - T_2 - km_1\, g = m_1\, w \tag{2}$$

and $$T_2 - km_2\, g = m_2\, w \tag{3}$$



The simultaneous solution of Eqs. (1), (2) and (3) yields,

$$w = g\,\frac{[\, m_0 - k\,(m_1 + m_2)\,]}{m_0 + m_1 + m_2}$$

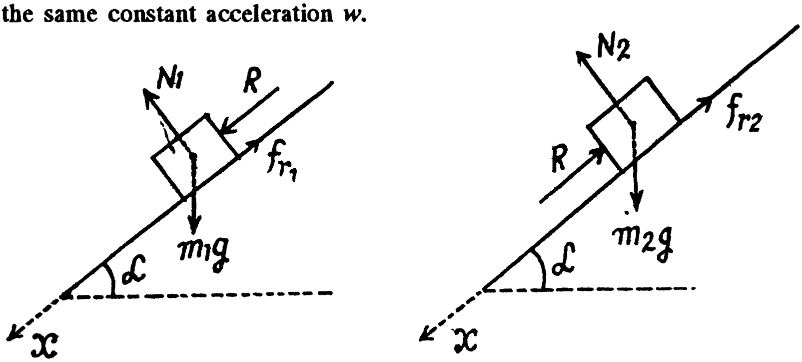
and $$T_2 = \frac{(1 + k)\, m_0}{m_0 + m_1 + m_2}\, m_2\, g$$

As the block m_0 moves down with acceleration w, so in vector form

$$\vec{w} = \frac{[m_0 - k\,(m_1 + m_2)\,]\,\vec{g}}{m_0 + m_1 + m_2}$$

1.61 Let us indicate the positive direction of x-axis along the incline (Fig.). Figures show the force diagram for the blocks.

Let, R be the force of interaction between the bars and they are obviously sliding down with the same constant acceleration w.



Newton's second law of motion in projection form along x-axis for the blocks gives :

$$m_1 g \sin\alpha - k_1 m_1 g \cos\alpha + R = m_1 w \qquad (1)$$

$$m_2 g \sin\alpha - R - k_2 m_2 g \cos\alpha = m_2 w \qquad (2)$$

Solving Eqs. (1) and (2) simultaneously, we get

$$w = g\sin\alpha - g\cos\alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} \text{ and}$$

$$R = \frac{m_1 m_2 (k_1 - k_2)\, g\cos\alpha}{m_1 + m_2} \qquad (3)$$

(b) when the blocks just slide down the plane, $w = 0$, so from Eqn. (3)

$$g\sin\alpha - g\cos\alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} = 0$$

$$\text{or,} \quad (m_1 + m_2)\sin\alpha = (k_1 m_1 + k_2 m_2)\cos\alpha$$

Hence $$\tan\alpha = \frac{(k_1 m_1 + k_2 m_2)}{m_1 + m_2}$$

1.62 Case 1. When the body is launched up :

Let k be the coefficeint of friction, u the velocity of projection and l the distance traversed along the incline. Retarding force on the block $= mg\sin\alpha + k\,mg\cos\alpha$ and hence the retardation $= g\sin\alpha + k\,g\cos\alpha$.

Using the equation of particle kinematics along the incline,

$$0 = u^2 - 2\,(g\sin\alpha + k\,g\cos\alpha)\,l$$

or, $$l = \frac{u^2}{2\,(g\sin\alpha + k\,g\cos\alpha)} \qquad (1)$$

and $$0 = u - (g\sin\alpha + k\,g\cos\alpha)\,t$$

or, $$u = (g\sin\alpha + k\,g\cos\alpha)\,t \qquad (2)$$

Using (2) in (1) $l = \frac{1}{2}(g\sin\alpha + k\,g\cos\alpha)\,t^2$ (3)

Case (2). When the block comes downward, the net force on the body $= mg\sin\alpha - km\,g\cos\alpha$ and hence its acceleration $= g\sin\alpha - k\,g\cos\alpha$

Let, t be the time required then,

$$l = \frac{1}{2}(g\sin\alpha - k\,g\cos\alpha)\,t'^2 \qquad (4)$$

From Eqs. (3) and (4)

$$\frac{t^2}{t'^2} = \frac{\sin\alpha - k\cos\alpha}{\sin\alpha + k\cos\alpha}$$

But $\frac{t}{t'} = \frac{1}{\eta}$ (according to the question),

Hence on solving we get

$$k = \frac{(\eta^2 - 1)}{(\eta^2 + 1)}\tan\alpha = 0{\cdot}16$$

1.63 At the initial moment, obviously the tension in the thread connecting m_1 and m_2 equals the weight of m_2.

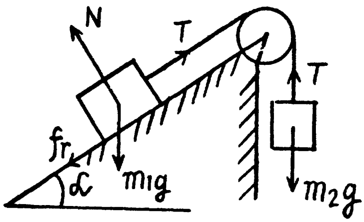
(a) For the block m_2 to come down or the block m_1 to go up, the conditions is

$$m_2 g - T \geq 0 \quad \text{and} \quad T - m_1 g \sin\alpha - fr \geq 0$$

where T is tension and f_r is friction which in the limiting case equals $km_1 g \cos\alpha$. Then

or $$m_2 g - m_1 \sin\alpha > k m_1 g \cos\alpha$$

or $$\frac{m_2}{m_1} > (k \cos\alpha + \sin\alpha)$$



(b) Similarly in the case

$$m_1 g \sin\alpha - m_2 g > fr_{lim}$$

or, $$m_1 g \sin\alpha - m_2 g > km_1 g \cos\alpha$$

or, $$\frac{m_2}{m_1} < (\sin\alpha - k\cos\alpha)$$

(c) For this case, neither kind of motion is possible, and fr need not be limiting.

Hence, $$(k\cos\alpha + \sin\alpha) > \frac{m_2}{m_1} > (\sin\alpha - k\cos\alpha)$$

1.64 From the conditions, obtained in the previous problem, first we will check whether the mass m_2 goes up or down.

Here, $m_2/m_1 = \eta > \sin\alpha + k\cos\alpha$, (substituting the values). Hence the mass m_2 will come down with an acceleration (say w). From the free body diagram of previous problem,

$$m_2 - g - T = m_2 w \tag{1}$$

and $$T - m_1 g \sin\alpha - k m_1 g \cos\alpha = m_1 w \tag{2}$$

Adding (1) and (2), we get,

$$m_2 g - m_1 g \sin\alpha - k m_1 g \cos\alpha = (m_1 + m_2) w$$

$$w = \frac{(m_2/m_1 - \sin\alpha - k\cos\alpha) g}{(1 + m_2/m_1)} = \frac{(\eta - \sin\alpha - k\cos\alpha) g}{1 + \eta}$$

Substituting all the values, $w = 0{\cdot}048\,g \approx 0{\cdot}05\,g$

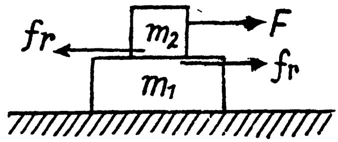
As m_2 moves down with acceleration of magnitude $w = 0.05\,g > 0$, thus in vector form acceleration of m_2:

$$\vec{w}_2 = \frac{(\eta - \sin\alpha - k\cos\alpha)\vec{g}}{1 + \eta} = 0.05\,\vec{g}.$$

1.65 Let us write the Newton's second law in projection form along positive x-axis for the plank and the bar

$$fr = m_1 w_1, \; fr = m_2 w_2 \tag{1}$$

At the initial moment, fr represents the static friction, and as the force F grows so does the friction force fr, but up to it's limiting value i.e. $fr = fr_{s(max)} = kN = k m_2 g$.



Unless this value is reached, both bodies moves as a single body with equal acceleration. But as soon as the force fr reaches the limit, the bar starts sliding over the plank i.e. $w_2 \geq w_1$.

Substituting here the values of w_1 and w_2 taken from Eq. (1) and taking into account that $f_r = k m_2 g$, we obtain, $(at - km_2 g)/m_2 \geq \frac{km_2}{m_1} g$, were the sign "=" corresponds to the moment $t = t_0$ (say)

Hence, $$t_0 = \frac{k g m_2 (m_1 + m_2)}{a m_1}$$

If $t \leq t_0$, then $w_1 = \frac{km_2 g}{m_1}$ (constant). and

$$w_2 = (at - km_2 g)/m_2$$

On this basis $w_1(t)$ and $w_2(t)$, plots are as shown in the figure of answersheet.

1.66 Let us designate the x-axis (Fig.) and apply $F_x = m w_x$ for body A :

$$mg \sin\alpha - k m g \cos\alpha = m w$$

or, $$w = g \sin\alpha - k g \cos\alpha$$

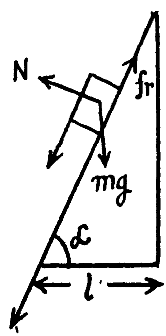
Now, from kinematical equation :

$$l \sec\alpha = 0 + (1/2) w t^2$$

or, $$t = \sqrt{2 l \sec\alpha/(\sin\alpha - k\cos\alpha) g}$$

$$= \sqrt{2 l/(\sin 2\alpha/2 - k\cos^2\alpha) g}$$

(using Eq. (1)).



for t_{min}, $$\frac{d\left(\frac{\sin 2\alpha}{2} - k\cos^2\alpha\right)}{d\alpha} = 0$$

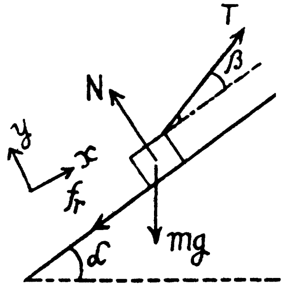
i.e. $$\frac{2\cos 2\alpha}{2} + 2k\cos\alpha\sin\alpha = 0$$

or, $$\tan 2\alpha = -\frac{1}{k} \Rightarrow \alpha = 49°$$

and putting the values of α, k and l in Eq. (2) we get $t_{min} = 1$s.

1.67 Let us fix the $x - y$ co-ordinate system to the wedge, taking the x – axis up, along the incline and the y – axis perpendicular to it (Fig.).

Now, we draw the free body diagram for the bar.



Let us apply Newton's second law in projection form along x and y axis for the bar :

$T\cos\beta - mg\sin\alpha - fr = 0$ (1)

$T\sin\beta + N - mg\cos\alpha = 0$

or, $N = mg\cos\alpha - T\sin\beta$ (2)

But $f_r = kN$ and using (2) in (1), we get

$$T = mg\sin\alpha + k\,mg\cos\alpha/(\cos\beta + k\sin\beta) \quad (3)$$

For T_{min} the value of $(\cos\beta + k\sin\beta)$ should be maximum

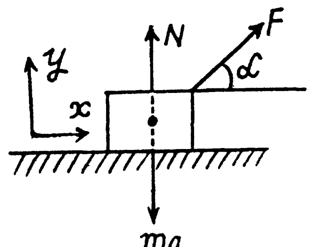
So, $$\frac{d(\cos\beta + k\sin\beta)}{d\beta} = 0 \text{ or } \tan\beta = k$$

Putting this value of β in Eq. (3) we get,

$$T_{min} = \frac{mg(\sin\alpha + k\cos\alpha)}{1/\sqrt{1+k^2} + k^2/\sqrt{1+k^2}} = \frac{mg(\sin\alpha + k\cos\alpha)}{\sqrt{1+k^2}}$$

1.68 First of all let us draw the free body diagram for the small body of mass m and indicate x – axis along the horizontal plane and y – axis, perpendicular to it, as shown in the figure. Let the block breaks off the plane at $t = t_0$ i.e. $N = 0$

So, $N = mg - at_0\sin\alpha = 0$



or, $$t_0 = \frac{mg}{a\sin\alpha} \quad (1)$$

From $F_x = m w_x$, for the body under investigation :

$m\,dv_x/dt = at\cos\alpha$; Integrating within the limits for $v(t)$

$$m\int_0^v dv_x = a\cos\alpha\int_0 t\,dt \quad \text{(using Eq. 1)}$$

So, $$v = \frac{ds}{dt} = \frac{a\cos\alpha}{2m}t^2 \quad (2)$$

Integrating, Eqn. (2) for $s(t)$

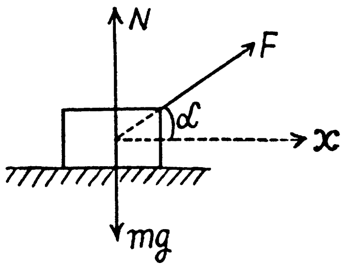
$$s = \frac{a\cos\alpha}{2m}\frac{t^3}{3} \quad (3)$$

Using the value of $t = t_0$ from Eq. (1), into Eqs. (2) and (3)

$$v = \frac{mg^2\cos\alpha}{2a\sin^2\alpha} \text{ and } s = \frac{m^2g^3\cos\alpha}{6a^2\sin^3\alpha}$$

1.69 Newton's second law of motion in projection form, along horizontal or x – axis i.e. $F_x = m\, w_x$ gives.

$$F \cos(as) = m\, v \frac{dv}{ds} \quad (\text{as } \alpha = as)$$



or, $\quad F \cos(as)\, ds = m\, v\, dv$

Integrating, over the limits for $v(s)$

$$\frac{F}{m}\int_0^{\infty} \cos(as)\, ds = \frac{v^2}{2}$$

or $\quad v = \sqrt{\dfrac{2F \sin\alpha}{m\, a}}$

$= \sqrt{2\, g \sin\alpha / 3\, a}$ (using $F = \dfrac{m\, g}{3}$)

which is the sought relationship.

1.70 From the Newton's second law in projection from :

For the bar,

$$T - 2\, kmg = (2m)\, w \qquad (1)$$

For the motor,

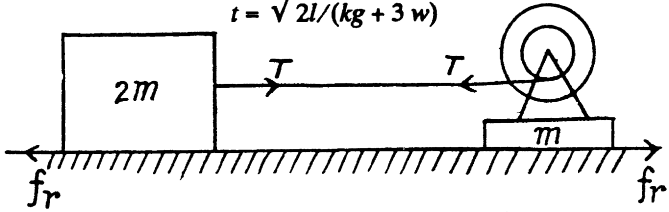
$$T - kmg = m\, w' \qquad (2)$$

Now, from the equation of kinematics in the frame of bar or motor :

$$l = \frac{1}{2}(w + w')\, t^2 \qquad (3)$$

From (1), (2) and (3) we get on eliminating T and w'

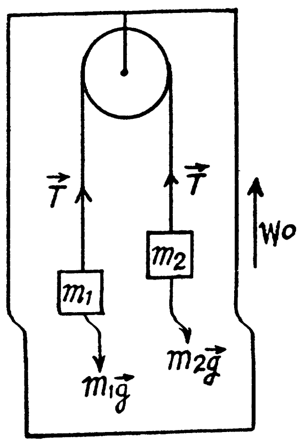
$$t = \sqrt{2l/(kg + 3\, w)}$$



1.71 Let us write Newton's second law in vector from $\vec{F} = m\,\vec{w}$, for both the blocks (in the frame of ground).

$$\vec{T} + m_1\,\vec{g} = m_1\,\vec{w}_1 \qquad (1)$$

$$\vec{T} + m_2\,\vec{g} = m_2\,\vec{w}_2 \qquad (2)$$



These two equations contain three unknown quantities $\vec{w}_1$, $\vec{w}_2$ and $\vec{T}$. The third equation is provided by the kinematic relationship between the accelerations :

$$\vec{w}_1 = \vec{w}_0 + \vec{w}', \quad \vec{w}_2 = \vec{w}_0 - \vec{w}' \qquad (3)$$

where $\vec{w}'$ is th acceleration of the mass m_1 with respect to the pulley or elevator car. Summing up termwise the left hand and the right-hand sides of these kinematical equations, we get

$$\vec{w}_1 + \vec{w}_2 = 2\,\vec{w}_0 \tag{4}$$

The simultaneous solution of Eqs.(1), (2) and (4) yields

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g} + 2\,m_2\,\vec{w}_0}{m_1 + m_2}$$

Using this result in Eq. (3), we get,

$$\vec{w}' = \frac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0) \quad \text{and} \quad \vec{T} = \frac{2\,m_1\,m_2}{m_1 + m_2}(\vec{w}_0 - \vec{g})$$

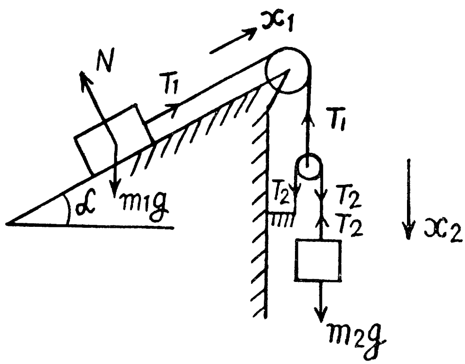
Using the results in Eq. (3) we get $\vec{w}' = \frac{m_1 - m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0)$

(b) obviously the force exerted by the pulley on the celing of the car

$$\vec{F} = -2\,\vec{T} = \frac{4\,m_1\,m_2}{m_1 + m_2}(\vec{g} - \vec{w}_0)$$

Note : *one could also solve this problem in the frame of elevator car.*

1.72 Let us write Newton's second law for both, bar 1 and body 2 in terms of projection having taken the positive direction of x_1 and x_2 as shown in the figure and assuming that body 2 starts sliding, say, upward along the incline



$$T_1 - m_1\,g\sin\alpha = m_1\,w_1 \tag{1}$$

$$m_2\,g - T_2 = m_2\,w \tag{2}$$

For the pulley, moving in vertical direction from the equation $F_x = m\,w_x$

$$2\,T_2 - T_1 = (m_p)\,w_1 = 0$$

(as mass of the pulley $m_p = 0$)

or

$$T_1 = 2\,T_2 \tag{3}$$

As the length of the threads are constant, the kinematical relationship of accelerations becomes

$$w = 2\,w_1 \tag{4}$$

Simultaneous solutions of all these equations yields :

$$w = \frac{2\,g\left(2\,\frac{m_2}{m_1} - \sin\alpha\right)}{\left(4\,\frac{m_2}{m_1} + 1\right)} = \frac{2\,g\,(2\,\eta - \sin\alpha)}{(4\,\eta + 1)}$$

As $\eta > 1$, w is directed vertically downward, and hence in vector form

$$\vec{w} = \frac{2\,\vec{g}\,(2\,\eta - \sin\alpha)}{4\,\eta + 1}$$

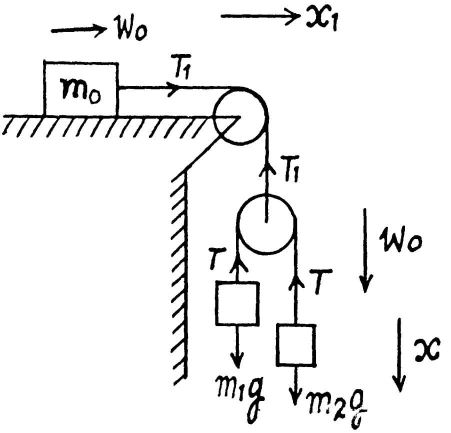
1.73 Let us write Newton's second law for masses m_1 and m_2 and moving pully in vertical direction along positive x – axis (Fig.) :

$$m_1 g - T = m_1 w_{1x} \quad (1)$$

$$m_2 g - T = m_2 w_{2x} \quad (2)$$

$$T_1 - 2T = 0 \text{ (as } m = 0)$$

or
$$T_1 = 2T \quad (3)$$



Again using Newton's second law in projection form for mass m_0 along positive x_1 direction (Fig.), we get

$$T_1 = m_0 w_0 \quad (4)$$

The kinematical relationship between the accelerations of masses gives in terms of projection on the x – axis

$$w_{1x} + w_{2x} = 2 w_0 \quad (5)$$

Simultaneous solution of the obtained five equations yields :

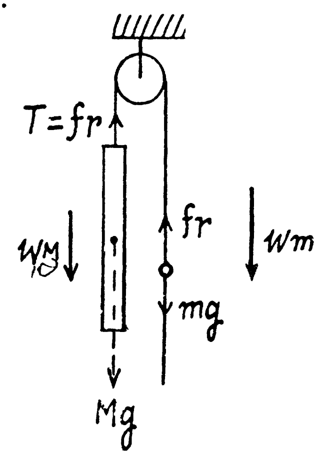
$$w_1 = \frac{[4 m_1 m_2 + m_0 (m_1 - m_2)] g}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

In vector form

$$\vec{w}_1 = \frac{[4 m_1 m_2 + m_0 (m_1 - m_2)] \vec{g}}{4 m_1 m_2 + m_0 (m_1 + m_2)}$$

1.74 As the thread is not tied with m, so if there were no friction between the thread and the ball m, the tension in the thread would be zero and as a result both bodies will have free fall motion. Obviously in the given problem it is the friction force exerted by the ball on the thread, which becomes the tension in the thread. From the condition or language of the problem $w_M > w_m$ and as both are directed downward so, relative acceleration of $M = w_M - w_m$ and is directed downward. Kinematical equation for the ball in the frame of rod in projection form along upward direction gives :

$$l = \frac{1}{2} (w_M - w_m) t^2 \quad (1)$$



Newton's second law in projection form along vertically down direction for both, rod and ball gives,

$$Mg - fr = M w_M \quad (2)$$

$$mg - fr = m w_m \quad (3)$$

Multiplying Eq. (2) by m and Eq. (3) by M and then subtracting Eq. (3) from (2) and after using Eq. (1) we get

$$fr = \frac{2 l M m}{(M - m) t^2}$$

1.75 Suppose, the ball goes up with accleration w_1 and the rod comes down with the acceleration w_2.

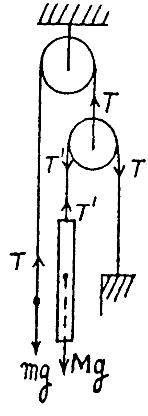
As the length of the thread is constant, $2\,w_1 = w_2$ (1)

From Newton's second law in projection form along vertically upward for the ball and vertically downward for the rod respectively gives,

$$T - mg = m\,w_1 \qquad (2)$$

and $$Mg - T' = M\,w_2 \qquad (3)$$

but $$T = 2\,T \quad \text{(because pulley is massless)} \qquad (4)$$



From Eqs. (1), (2), (3) and (4)

$$w_1 = \frac{(2M - m)\,g}{m + 4M} = \frac{(2 - \eta)\,g}{\eta + 4} \text{ (in upward direction)}$$

and $$w_2 = \frac{2\,(2 - \eta)\,g}{(\eta + 4)} \text{ (downwards)}$$

From kinematical equation in projection form, we get

$$l = \frac{1}{2}(w_1 + w_2)\,t^2$$

as, w_1 and w_2 are in the opposite direction.

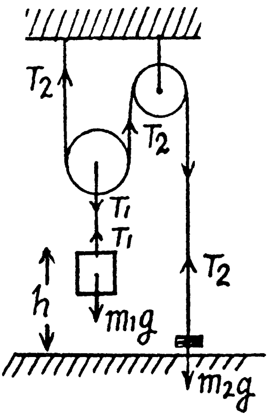
Putting the values of w_1 and w_2, the sought time becomes

$$t = \sqrt{2\,l\,(\eta + 4)\,/\,3\,(2 - \eta)\,g} = 1{\cdot}4\text{ s}$$

1.76 Using Newton's second law in projection form along x – axis for the body 1 and along negative x – axis for the body 2 respectively, we get

$$m_1\,g - T_1 = m_1\,w_1 \qquad (1)$$

$$T_2 - m_2\,g = m_2\,w_2 \qquad (2)$$



For the pulley lowering in downward direction from Newton's law along x axis,

$$T_1 - 2\,T_2 = 0 \text{ (as pulley is mass less)}$$

or, $$T_1 = 2\,T_2 \qquad (3)$$

As the length of the thread is constant so,

$$w_2 = 2\,w_1 \qquad (4)$$

The simultaneous solution of above equations yields,

$$w_2 = \frac{2\,(m_1 - 2m_2)\,g}{4\,m_2 + m_1} = \frac{2\,(\eta - 2)}{\eta + 4} \quad \left(\text{as } \frac{m_1}{m_2} = \eta\right) \qquad (5)$$

Obviously during the time interval in which the body 1 comes to the horizontal floor covering the distance h, the body 2 moves upward the distance $2h$. At the moment when the body 2 is at the height $2h$ from the floor its velocity is given by the expression :

$$v_2^2 = 2\,w_2\,(2h) = 2\left[\frac{2\,(\eta - 2)\,g}{\eta + 4}\right]2h = \frac{8\,h\,(\eta - 2)\,g}{\eta + 4}$$

After the body m_1 touches the floor the thread becomes slack or the tension in the thread zero, thus as a result body 2 is only under gravity for it's subsequent motion.

Owing to the velocity v_2 at that moment or at the height $2h$ from the floor, the body 2 further goes up under gravity by the distance,

$$h' = \frac{v_2^2}{2g} = \frac{4h(\eta - 2)}{\eta + 4}$$

Thus the sought maximum height attained by the body 2 :

$$H = 2h + h' = 2h + \frac{4h(\eta - 2)}{(\eta + 4)} = \frac{6\eta h}{\eta + 4}$$

1.77 Let us draw free body diagram of each body, i.e. of rod A and of wedge B and also draw the kinemetical diagram for accelerations, after analysing the directions of motion of A and B. Kinematical relationship of accelarations is :

$$\tan\alpha = \frac{w_A}{w_B} \tag{1}$$

Let us write Newton's second law for both bodies in terms of projections having taken positive directions of y and x axes as shown in the figure.

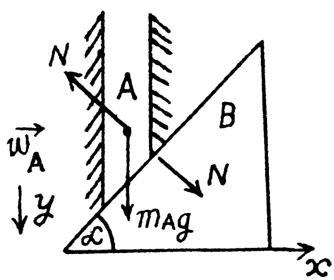
$$m_A g - N\cos\alpha = m_A w_A \tag{2}$$

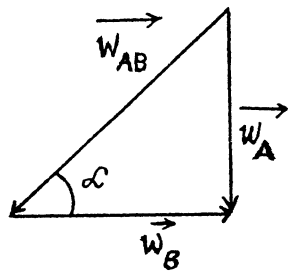
and $$N\sin\alpha = m_B w_B \tag{3}$$

Simultaneous solution of (1), (2) and (3) yields :

$$w_A = \frac{m_A g \sin\alpha}{m_A \sin\alpha + m_B \cot\alpha\cos\alpha} = \frac{g}{(1 + \eta\cot^2\alpha)} \text{ and}$$

$$w_B = \frac{w_A}{\tan\alpha} = \frac{g}{(\tan\alpha + \eta\cot\alpha)}$$





Note : *We may also solve this problem using conservation of mechanical energy instead of Newton's second law.*

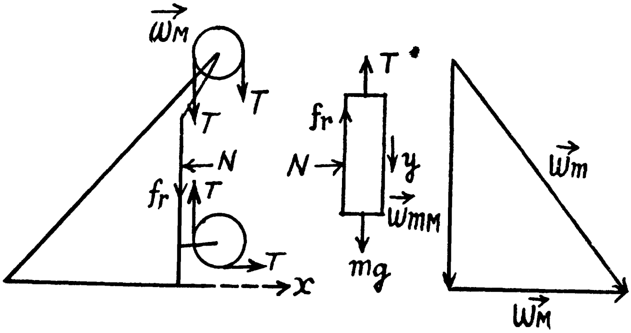
1.78 Let us draw free body diagram of each body and fix the coordinate system, as shown in the figure. After analysing the motion of M and m on the basis of force diagrams, let us draw the kinematical diagram for accelerations (Fig.).

As the length of threads are constant so,

$ds_{mM} = ds_M$ and as $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions that why

$$|\vec{w}_{mM}| = |\vec{w}_M| = w \text{ (say) and}$$

$$\vec{w}_{mM} \uparrow\uparrow \vec{v}_M \text{ and } \vec{w}_M \uparrow\uparrow \vec{v}_M$$



As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$

so, from the triangle law of vector addition

$$w_m = \sqrt{2}\, w \quad (1)$$

From the Eq. $F_x = m\, w_x$, for the wedge and block :

$$T - N = Mw, \quad (2)$$

and

$$N = mw \quad (3)$$

Now, from the Eq. $F_y = mw_y$, for the block

$$mg - T - kN = mw \quad (4)$$

Simultaneous solution of Eqs. (2), (3) and (4) yields :

$$w = \frac{mg}{(km + 2m + M)} = \frac{g}{(k + 2 + M/m)}$$

Hence using Eq. (1)

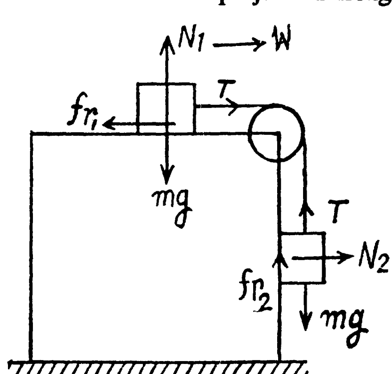
$$w_m = \frac{g\sqrt{2}}{(2 + k + M/m)}$$

1.79 Bodies 1 and 2 will remain at rest with repect to bar A for $w_{min} \le w \le w_{max}$, where w_{min} is the sought minimum acceleration of the bar. Beyond these limits there will be a relative motion between bar and the bodies. For $0 \le w \le w_{min}$, the tendency of body 1 in relation to the bar A is to move towards right and is in the opposite sense for $w \ge w_{max}$. On the basis of above argument the static friction on 2 by A is directed upward and on 1 by A is directed towards left for the purpose of calculating w_{min}.

Let us write Newton's second law for bodies 1 and 2 in terms of projection along positive x – axis (Fig.).

$T - fr_1 = m\,w$ or, $fr_1 = T - mw$ (1)

$N_2 = mw$ (2)



As body 2 has no acceleration in vertical direction, so

$fr_2 = mg - T$ (3)

From (1) and (3)

$(fr_1 + fr_2) = m\,(g - w)$ (4)

But $fr_1 + fr_2 \le k\,(N_1 + N_2)$

or $fr_1 + fr_2 \le k\,(mg + mw)$ (5)

From (4) and (5)

$$m(g-w) \le mk(g+w), \text{ or } w \ge \frac{g(1-k)}{(1+k)}$$

Hence
$$w_{min} = \frac{g(1-k)}{(1+k)}$$

1.80 On the basis of the initial argument of the solution of 1.79, the tendency of bar 2 with respect to 1 will be to move up along the plane.

Let us fix $(x-y)$ coordinate system in the frame of ground as shown in the figure.

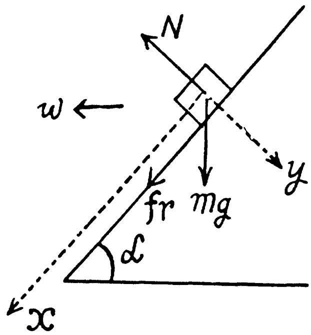
From second law of motion in projection form along y and x axes :

$m g \cos\alpha - N = m w \sin\alpha$

or, $N = m(g\cos\alpha - w\sin\alpha)$ (1)

$m g \sin\alpha + fr = m w \cos\alpha$

or, $fr = m(w\cos\alpha - g\sin\alpha)$ (2)



but $fr \le kN$, so from (1) and (2)

$(w\cos\alpha - g\sin\alpha) \le k(g\cos\alpha + w\sin\alpha)$

or, $w(\cos\alpha - k\sin\alpha) \le g(k\cos\alpha + \sin\alpha)$

or, $$w \le g\frac{(\cos\alpha + \sin\alpha)}{\cos\alpha - k\sin\alpha},$$

So, the sought maximum acceleration of the wedge :

$$w_{max} = \frac{(k\cos\alpha + \sin\alpha)g}{\cos\alpha - k\sin\alpha} = \frac{(k\cot\alpha + 1)g}{\cot\alpha - k} \quad \text{where } \cot\alpha > k$$

1.81 Let us draw the force diagram of each body, and on this basis we observe that the prism moves towards right say with an acceleration $\vec{w}_1$ and the bar 2 of mass m_2 moves down the plane with respect to 1, say with acceleration $\vec{w}_{21}$, then, $\vec{w}_2 = \vec{w}_{21} + \vec{w}_1$ (Fig.)

Let us write Newton's second law for both bodies in projection form along positive y_2 and x_1 axes as shown in the Fig.

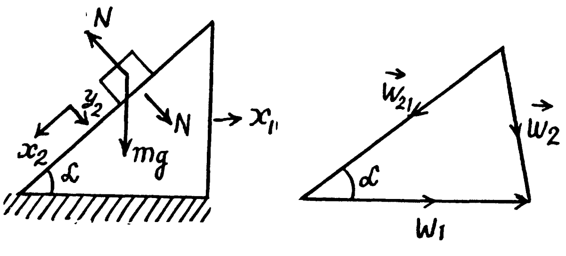
$$m_2 g\cos\alpha - N = m_2 w_{2(y_2)} = m_2\left[w_{21(y_2)} + w_{1(y_2)}\right] = m_2\left[0 + w_1\sin\alpha\right]$$

or, $$m_2 g\cos\alpha - N = m_2 w_1 \sin\alpha \quad (1)$$

and $$N\sin\alpha = m_1 w_1 \quad (2)$$

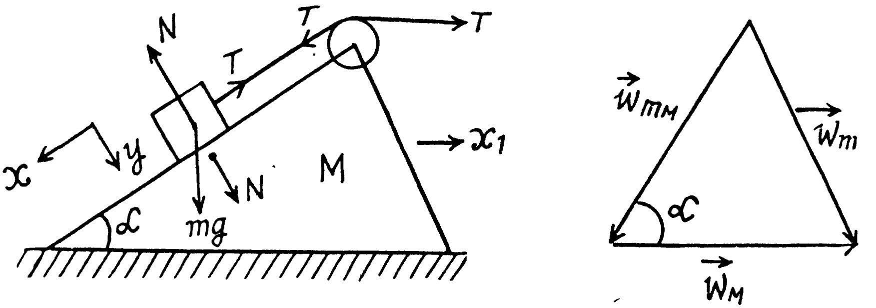
Solving (1) and (2), we get

$$w_1 = \frac{m_2 g\sin\alpha\cos\alpha}{m_1 + m_2\sin^2\alpha} = \frac{g\sin\alpha\cos\alpha}{(m_1/m_2) + \sin^2\alpha}$$



1.82 To analyse the kinematic relations between the bodies, sketch the force diagram of each body as shown in the figure.

On the basis of force diagram, it is obvious that the wedge M will move towards right and the block will move down along the wedge. As the length of the thread is constant, the distance travelled by the block on the wedge must be equal to the distance travelled by the wedge on the floor. Hence $d s_{mM} = d s_M$. As $\vec{v}_{mM}$ and $\vec{v}_M$ do not change their directions and acceleration that's why $\vec{w}_{mM} \uparrow\uparrow \vec{v}_{mM}$ and $\vec{w}_M \uparrow\uparrow \vec{v}_M$ and $w_{mM} = w_M = w$ (say) and accordingly the diagram of kinematical dependence is shown in figure.



As $\vec{w}_m = \vec{w}_{mM} + \vec{w}_M$, so from triangle law of vector addition.

$$w_m = \sqrt{w_M^2 + w_{mM}^2 - 2\, w_{mM}\, w_M \cos\alpha} = w\sqrt{2(1-\cos\alpha)} \quad (1)$$

From $F_x = m\, w_x$, (for the wedge),

$$T - T\cos\alpha + N\sin\alpha = Mw \quad (2)$$

For the bar m let us fix $(x-y)$ coordinate system in the frame of ground Newton's law in projection form along x and y axes (Fig.) gives

$$mg\sin\alpha - T = m\, w_{m(x)} = m\left[w_{mM(x)} + w_{M(x)}\right]$$

$$= m\left[w_{mM} + w_M\cos(\pi-\alpha)\right] = m\, w\,(1-\cos\alpha) \quad (3)$$

$$m\, g\cos\alpha - N = m\, w_{m(y)} = m\left[w_{mM(y)} + w_{M(y)}\right] = m\,[\,0 + w\sin\alpha\,] \quad (4)$$

Solving the above Eqs. simultaneously, we get

$$w = \frac{m\, g\sin\alpha}{M + 2m\,(1-\cos\alpha)}$$

Note : *We can study the motion of the block m in the frame of wedge also, alternately we may solve this problem using conservation of mechanical energy.*

1.83 Let us sketch the diagram for the motion of the particle of mass m along the circle of radius R and indicate x and y axis, as shown in the figure.

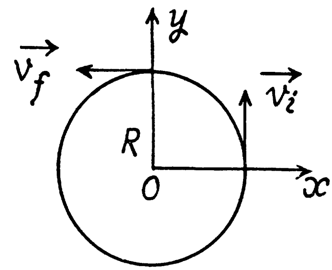
(a) For the particle, change in momentum $\Delta\vec{p} = mv(-\vec{i}) - m\, v(\vec{j})$

so, $|\Delta\vec{p}| = \sqrt{2}\, m\, v$

and time taken in describing quarter of the circle,

$$\Delta t = \frac{\pi R}{2v}$$

Hence, $<\vec{F}> = \frac{|\Delta \vec{p}|}{\Delta t} = \frac{\sqrt{2}\, mv}{\pi R/2v} = \frac{2\sqrt{2}\, m v^2}{\pi R}$



(b) In this case

$\vec{p_i} = 0$ and $\vec{p_f} = m w_t t (-\vec{i})$,

so $|\Delta \vec{p}| = m w_t t$

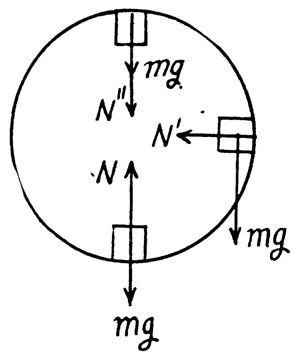
Hence, $|<\vec{F}>| = \frac{|\Delta \vec{p}|}{t} = m w_t$

1.84 While moving in a loop, normal reaction exerted by the flyer on the loop at different points and uncompensated weight if any contribute to the weight of flyer at those points.

(a) When the aircraft is at the lowermost point, Newton's second law of motion in projection form $F_n = m w_n$ gives

$$N - mg = \frac{m v^2}{R}$$

or, $N = mg + \frac{m v^2}{R} = 2{\cdot}09$ kN



(b) When it is at the upper most point, again from $F_n = m w_n$ we get

$$N'' + mg = \frac{mv^2}{R}$$

$$N'' = \frac{m v^2}{R} - m g = 0{\cdot}7 \text{ kN}$$

(c) When the aircraft is at the middle point of the loop, again from $F_n = m w_n$

$$N' = \frac{m v^2}{R} = 1{\cdot}4 \text{ kN}$$

The uncompensated weight is mg. Thus effective weight $= \sqrt{N'^2 + m^2 g^2} = 1{\cdot}56$ kN acts obliquely.

1.85 Let us depict the forces acting on the small sphere m, (at an arbitrary position when the thread makes an angle θ from the vertical) and write equation $\vec{F} = m \vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$. From $F_t = m w_t$, we have

$$mg \sin \theta = m \frac{dv}{dt}$$

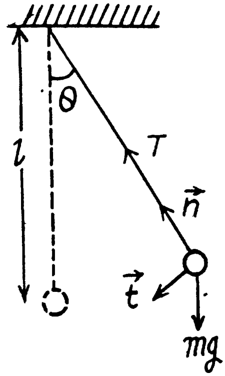
$$= m \frac{v\, dv}{ds} = m \frac{v\, dv}{l(-d\theta)}$$

(as vertical is refrence line of angular position) or $$v\,dv = -gl\sin\theta\,d\theta$$

Integrating both the sides :

$$\int_0^v v\,dv = -g\,l\int_{\pi/2}^{\theta}\sin\theta\,d\theta$$

or, $$\frac{v^2}{2} = g\,l\cos\theta$$



Hence $\frac{v^2}{l} = 2\,g\cos\theta = w_n$ (1)

(Eq. (1) can be easily obtained by the conservation of mechanical energy).

From $F_n = m\,w_n$

$$T - m\,g\cos\theta = \frac{m\,v^2}{l}$$

Using (1) we have

$$T = 3\,mg\cos\theta \quad (2)$$

Again from the Eq. $F_t = m\,w_t$:

$mg\sin\theta = m\,w_t$ or $w_t = g\sin\theta$ (3)

Hence $w = \sqrt{w_t^2 + w_n^2} = \sqrt{(g\sin\theta)^2 + (2g\cos\theta)^2}$ (using 1 and 3)

$$= g\sqrt{1 + 3\cos^2\theta}$$

(b) Vertical component of velocity, $v_y = v\sin\theta$

So, $$v_y^2 = v^2\sin^2\theta = 2g\,l\cos\theta\sin^2\theta \text{ (using 1)}$$

For maximum v_y or v_y^2, $\frac{d(\cos\theta\sin^2\theta)}{d\theta} = 0$

which yields $$\cos\theta = \frac{1}{\sqrt{3}}$$

Therefore from (2) $T = 3mg\frac{1}{\sqrt{3}} = \sqrt{3}\,mg$

(c) We have $\vec{w} = w_t\,\hat{u}_t + w_n\,\hat{u}_n$ thus $w_y = w_{t(y)} + w_{n(y)}$

But in accordance with the problem $w_y = 0$

So, $$w_{t(y)} + w_{n(y)} = 0$$

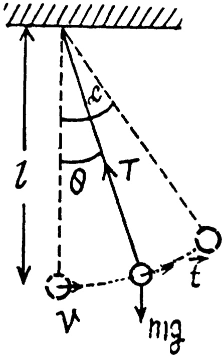
or, $$g\sin\theta\sin\theta + 2g\cos^2\theta\,(-\cos\theta) = 0$$

or, $$\cos\theta = \frac{1}{\sqrt{3}} \quad \text{or,} \quad \theta = 54{\cdot}7°$$

1.86 The ball has only normal acceleration at the lowest position and only tangential acceleration at any of the extreme position. Let v be the speed of the ball at its lowest position and l be the length of the thread, then according to the problem

$$\frac{v^2}{l} = g \sin \alpha \qquad (1)$$

where α is the maximum deflection angle



From Newton's law in projection form : $F_t = mw_t$

$$-mg \sin \theta = mv \frac{dv}{l\, d\theta}$$

or, $\quad -g\, l \sin \theta\, d\theta = v\, dv$

On integrating both the sides within their limits.

$$-gl \int_0^{\alpha} \sin \theta\, d\theta = \int_v^0 v\, dv$$

or, $\quad v^2 = 2gl\,(1 - \cos \alpha) \qquad (2)$

Note : Eq. (2) can easily be obtained by the conservation of mechanical energy of the ball in the uniform field of gravity.

From Eqs. (1) and (2) with $\theta = \alpha$

$$2gl\,(1 - \cos \alpha) = lg \cos \alpha$$

or, $\quad \cos \alpha = \frac{2}{3}$ so, $\alpha = 53°$

1.87 Let us depict the forces acting on the body A (which are the force of gravity $m\vec{g}$ and the normal reaction $\vec{N}$) and write equation $\vec{F} = m\vec{w}$ via projection on the unit vectors $\hat{u}_t$ and $\hat{u}_n$ (Fig.)

From $F_t = mw_t$

$$mg \sin \theta = m\frac{dv}{dt}$$

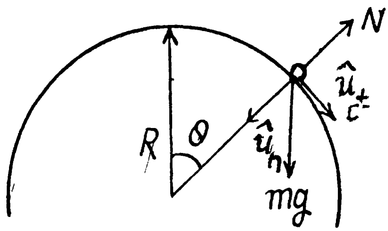
$$= m\frac{v\, dv}{ds} = m\frac{v\, dv}{R\, d\theta}$$

or, $\quad gR \sin \theta\, d\theta = v\, dv$

Integrating both side for obtaining $v\,(\theta)$

$$\int_0^{\theta} gR \sin \theta\, d\theta = \int_0^{v} v\, dv$$

or, $\quad v^2 = 2\,gR\,(1 - \cos \theta) \qquad (1)$



From $F_n = mw_n$

$$mg \cos \theta - N = m\frac{v^2}{R} \qquad (2)$$

At the moment the body loses contact with the surface, $N = 0$ and therefore the Eq. (2) becomes

$$v^2 = gR \cos \theta \qquad (3)$$

where v and θ correspond to the moment when the body loses contact with the surface. Solving Eqs. (1) and (3) we obtain $\cos\theta = \frac{2}{3}$ or, $\theta = \cos^{-1}(2/3)$ and $v = \sqrt{2gR/3}$.

1.88 At first draw the free body diagram of the device as, shown. The forces, acting on the sleeve are it's weight, acting vertically downward, spring force, along the length of the spring and normal reaction by the rod, perpendicular to its length.

Let F be the spring force, and Δl be the elongation.

From, $F_n = mw_n$:

$$N\sin\theta + F\cos\theta = m\,\omega^2 r \tag{1}$$

where $r\cos\theta = (l_0 + \Delta l)$.

Similarly from $F_t = mw_t$

$$N\cos\theta - F\sin\theta = 0 \quad \text{or,} \quad N = F\sin\theta/\cos\theta \tag{2}$$

From (1) and (2)

$$F(\sin\theta/\cos\theta)\cdot\sin\theta + F\cos\theta = m\,\omega^2 r$$

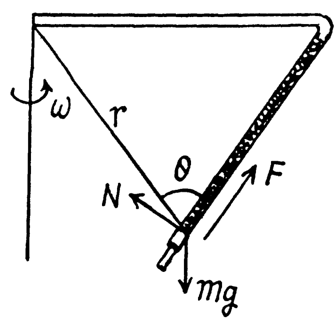
$$= m\,\omega^2(l_0 + \Delta l)/\cos\theta$$

On putting $F = \kappa\,\Delta\,l$,

$$\kappa\,\Delta\,l\sin^2\theta + \kappa\,\Delta\,l\cos^2\theta = m\,\omega^2(l_0 + \Delta\,l)$$

on solving, we get,

$$\Delta l = m\,\omega^2\frac{l_0}{\kappa - m\,\omega^2} = \frac{l_0}{(\kappa/m\,\omega^2 - 1)}$$



and it is independent of the direction of rotation.

1.89 According to the question, the cyclist moves along the circular path and the centripetal force is provided by the frictional force. Thus from the equation $F_n = m\,w_n$

$$fr = \frac{m\,v^2}{r} \quad \text{or} \quad kmg = \frac{m\,v^2}{r}$$

or

$$k_0\left(1 - \frac{r}{R}\right)g = \frac{v^2}{r} \quad \text{or} \quad v^2 = k_0(r - r^2/R)\,g \tag{1}$$

For v_{max}, we should have $\dfrac{d\left(r - \dfrac{r^2}{R}\right)}{dr} = 0$

or, $$1 - \frac{2r}{R} = 0, \quad \text{so } r = R/2$$

Hence, $v_{max} = \frac{1}{2}\sqrt{k_0 gR}$

1.90 As initial velocity is zero thus

$$v^2 = 2\,w_t\,s \tag{1}$$

As $w_t > 0$ the speed of the car increases with time or distance. Till the moment, sliding starts, the static friction provides the required centripetal acceleration to the car.

Thus $$fr = mw, \quad \text{but} \quad fr \le kmg$$

So, $$w^2 \le k^2 g^2 \text{ or, } w_t^2 + \frac{v^2}{R} \le k^2 g^2$$

or, $$v^2 \le (k^2 g^2 - w_t^2) R$$

Hence $$v_{max} = \sqrt{(k^2 g^2 - w_t^2) R}$$

so, from Eqn. (1), the sought distance $s = \frac{v_{max}^2}{2 w_t} = \frac{1}{2}\sqrt{\left(\frac{kg}{w_t}\right)^2 - 1} = 60 \text{ m}.$

1.91 Since the car follows a curve, so the maximum velocity at which it can ride without sliding at the point of minimum radius of curvature is the sought velocity and obviously in this case the static friction between the car and the road is limiting.

Hence from the equation $F_n = mw$

$$kmg \ge \frac{m v^2}{R} \quad \text{or} \quad v \le \sqrt{kRg}$$

so $$v_{max} = \sqrt{k R_{min} g}\,. \tag{1}$$

We know that, radius of curvature for a curve at any point (x, y) is given as,

$$R = \left| \frac{[1 + (dy/dx)^2]^{3/2}}{(d^2 y)/dx^2} \right| \tag{2}$$

For the given curve,

$$\frac{dy}{dx} = \frac{a}{\alpha}\cos\left(\frac{x}{\alpha}\right) \text{ and } \frac{d^2 y}{dx^2} = \frac{-a}{\alpha^2}\sin\frac{x}{\alpha}$$

Substituting this value in (2) we get,

$$R = \frac{[1 + (a^2/\alpha^2)\cos^2(x/\alpha)]^{3/2}}{(a/\alpha^2)\sin(x/\alpha)}$$

For the minimum R, $\frac{x}{\alpha} = \frac{\pi}{2}$

and therefore, corresponding radius of curvature

$$R_{min} = \frac{\alpha^2}{a} \tag{3}$$

Hence from (1) and (2)

$$v_{max} = \alpha\sqrt{kg/a}$$

1.92 The sought tensile stress acts on each element of the chain. Hence divide the chain into small, similar elements so that each element may be assumed as a particle. We consider one such element of mass dm, which subtends angle $d\alpha$ at the centre. The chain moves along a circle of known radius R with a known angular speed ω and certain forces act on it. We have to find one of these forces.

From Newton's second law in projection form, $F_x = mw_x$ we get

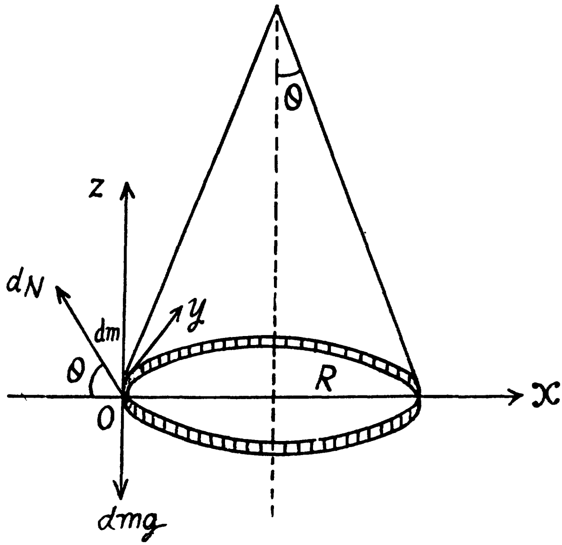
$$2T\sin(d\alpha/2) - dN\cos\theta = dm\,\omega^2 R$$

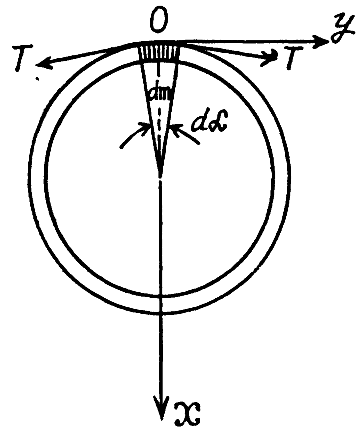
and from $F_Z = mw_Z$ we get

$$dN\sin\theta = g\,dm$$

Then putting $dm = m\,d\alpha/2\pi$ and $\sin(d\alpha/2) = d\alpha/2$ and solving, we get,

$$T = \frac{m(\omega^2 R + g\cot\theta)}{2\pi}$$





1.93 Let, us consider a small element of the thread and draw free body diagram for this element.

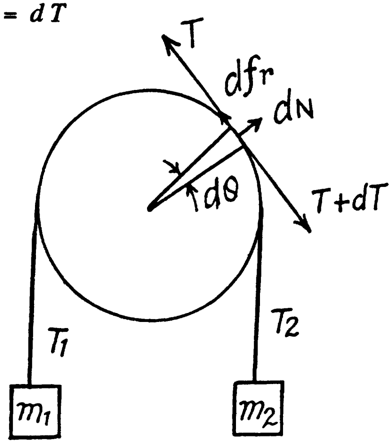
(a) Applying Newton's second law of motion in projection form, $F_n = mw_n$ for this element,

$(T + dT)\sin(d\theta/2) + T\sin(d\theta/2) - dN = dm\,\omega^2 R = 0$

or, $\quad 2T\sin(d\theta/2) = dN$, [negelecting the term$(dT\sin d\theta/2)$]

or, $$T\,d\theta = dN, \text{ as } \sin\frac{d\theta}{2} = \frac{d\theta}{2} \qquad (1)$$

Also, $$d\,fr = k\,dN = (T + dT) - T = dT \qquad (2)$$



From Eqs. (1) and (2),

$k\,T\,d\theta = dT$ or $\dfrac{dT}{T} = k\,d\theta$

In this case $Q = \pi$ so,

or, $\quad$ or, $\ln\dfrac{T_2}{T_1} = k\pi \qquad (3)$

So, $\quad k = \dfrac{1}{\pi}\ln\dfrac{T_2}{T_1} = \dfrac{1}{\pi}\ln\eta_0$

as $\quad \dfrac{T_2}{T_1} = \dfrac{m_2 g}{m_1 g} = \dfrac{m_2}{m_1} = \eta_0$

(b) When $\dfrac{m_2}{m_1} = \eta$, which is greater than η_0, the blocks will move with same value of acceleration. (say w) and clearly m_2 moves downward. From Newton's second law in projection form (downward for m_2 and upward for m_1) we get :

$$m_2 g - T_2 = m_2 w \qquad (4)$$

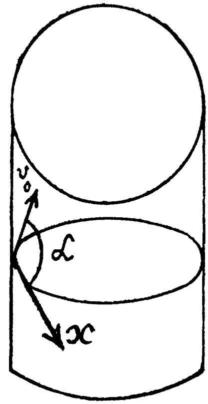
and $$T_1 - m_1 g = m_1 w \qquad (5)$$

Also $$\frac{T_2}{T_1} = \eta_0 \tag{6}$$

Simultaneous solution of Eqs. (4), (5) and (6) yields :

$$w = \frac{(m_2 - \eta_0 m_1) g}{(m_2 + \eta_0 m_1)} = \frac{(\eta - \eta_0)}{(\eta + \eta_0)} g \left(\text{as } \frac{m_2}{m_1} = \eta \right)$$

1.94 The force with which the cylinder wall acts on the particle will provide centripetal force necessary for the motion of the particle, and since there is no acceleration acting in the horizontal direction, horizontal component of the velocity will remain constant througout the motion.



So $$v_x = v_0 \cos \alpha$$

Using, $F_n = m\, w_n$, for the particle of mass m,

$$N = \frac{m v_x^2}{R} = \frac{m v_0^2 \cos^2 \alpha}{R},$$

which is the required normal force.

1.95 Obviously the radius vector describing the position of the particle relative to the origin of coordinate is

$$\vec{r} = x\vec{i} + y\vec{j} = a \sin \omega t\, \vec{i} + b \cos \omega t\, \vec{j}$$

Differentiating twice with respect the time :

$$\vec{w} = \frac{d^2 \vec{r}}{d t^2} = -\omega^2 (a \sin \omega t\, \vec{i} + b \cos \omega t\, \vec{j}) = -\omega^2 \vec{r} \tag{1}$$

Thus $$\vec{F} = m \vec{w} = -m \omega^2 \vec{r}$$

1.96 (a) We have $$\Delta \vec{p} = \int \vec{F} dt$$

$$= \int_0^t m \vec{g} dt = m \vec{g} t \tag{1}$$

(b) Using the solution of problem 1.28 (b), the total time of motion, $\tau = -\frac{2(\vec{v}_0 \cdot \vec{g})}{g^2}$

Hence using $t = \tau$ in (1)

$$|\Delta \vec{p}| = mg\tau$$

$$= -2m(\vec{v}_0 \cdot \vec{g})/g \quad (\vec{v}_0 \cdot \vec{g} \text{ is -ve})$$

1.97 From the equation of the given time dependence force $\vec{F} = \vec{a}\, t(\tau - t)$ at $t = \tau$, the force vanishes,

(a) Thus $$\Delta \vec{p} = \vec{p} = \int_0^\tau \vec{F} dt$$

or, $$\vec{p} = \int_0^{\tau} \vec{a}\, t\,(\tau - t)\, dt \; \frac{\vec{a}\,\tau^3}{6}$$

but $$\vec{p} = m\,\vec{v} \text{ so } \vec{v} = \frac{\vec{a}\,\tau^3}{6m}$$

(b) Again from the equation $\vec{F} = m\,\vec{w}$

$$\vec{a}t\,(\tau - t) = m\,\frac{d\,\vec{v}}{dt}$$

or, $$\vec{a}\,(t\,\tau - t^2)\, dt = m d\,\vec{v}$$

Integrating within the limits for $\vec{v}(t)$,

$$\int_0^{t} \vec{a}(t\,\tau - t^2)\, dt = m \int_0^{\vec{v}} d\,\vec{v}$$

or, $$\vec{v} = \frac{\vec{a}}{m}\left(\frac{\tau\, t^2}{2} - \frac{t^3}{3}\right) = \frac{\vec{a}\,t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right)$$

Thus $$v = \frac{a\,t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) \text{ for } t \le \tau$$

Hence distance covered during the time interval $t = \tau$,

$$s = \int_0^{\tau} v\, dt$$

$$= \int_0^{\tau} \frac{a\,t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) dt = \frac{a}{m}\,\frac{\tau^4}{12}$$

1.98 We have $F = F_0 \sin \omega t$

or $$m\,\frac{d\,\vec{v}}{dt} = \vec{F}_0 \sin \omega t \text{ or } m\, d\,\vec{v} = \vec{F}_0 \sin \omega t\, dt$$

On integrating,

$$\vec{mv} = \frac{-\vec{F}_0}{\omega} \cos \omega t + C, \text{ (where } C \text{ is integration constant)}$$

When $$t = 0,\ v = 0, \text{ so } C = \frac{\vec{F}_0}{m\omega}$$

Hence, $\vec{v} = \dfrac{-\vec{F}_0}{m\,\omega} \cos \omega t + \dfrac{\vec{F}_0}{m\,\omega}$

As $|\cos \omega t \le 1$ so, $v = \dfrac{F_0}{m\,\omega}(1 - \cos \omega t)$

Thus
$$s = \int_0^t v\,dt$$

$$= \frac{F_0 t}{m\,\omega} - \frac{F_0 \sin \omega t}{m\,\omega^2} = \frac{F_0}{m\,\omega^2}(\omega t - \sin \omega t)$$

(Figure in the answer sheet).

1.99 According to the problem, the force acting on the particle of mass m is, $\vec{F} = \vec{F}_0 \cos \omega t$

So,
$$m\frac{d\vec{v}}{dt} = \vec{F}_0 \cos \omega t \quad \text{or} \quad d\vec{v} = \frac{\vec{F}_0}{m} \cos \omega t\, dt$$

Integrating, within the limits.
$$\int_0^{\vec{v}} d\vec{v} = \frac{\vec{F}_0}{m}\int_0^t \cos \omega t\, dt \quad \text{or} \quad \vec{v} = \frac{\vec{F}_0}{m\,\omega} \sin \omega t$$

It is clear from equation (1), that after starting at $t = 0$, the particle comes to rest fro the first time at $t = \frac{\pi}{\omega}$.

From Eq. (1), $v = |\vec{v}| = \frac{F_0}{m\omega} \sin \omega t$ for $t \le \frac{\pi}{\omega}$ (2)

Thus during the time interval $t = \pi/\omega$, the sought distance
$$s = \frac{F_0}{mw}\int_0^{\pi/\omega} \sin \omega t\, dt = \frac{2F}{m\,\omega^2}$$

From Eq. (1)
$$v_{max} = \frac{F_0}{m\,\omega} \text{ as } |\sin \omega t| \le 1$$

1.100 (a) From the problem $\vec{F} = -r\vec{v}$ so $m\frac{d\vec{v}}{dt} = -r\vec{v}$

Thus
$$m\frac{dv}{dt} = -rv \text{ [as } d\vec{v} \uparrow\downarrow \vec{v}\text{]}$$

or,
$$\frac{dv}{v} = -\frac{r}{m}dt$$

On integrating
$$\ln v = -\frac{r}{m}t + C$$

But at
$$t = 0,\ v = v_0, \text{ so, } C = \ln v_0$$

or,
$$\ln\frac{v}{v_0} = -\frac{r}{m}t \quad \text{or,} \quad v = v_0 e^{-\frac{r}{m}t}$$

Thus for
$$t \to \infty,\ v = 0$$

(b) We have $m\frac{dv}{dt} = -rv$ so $dv = \frac{-r}{m}ds$

Integrating within the given limits to obtain $v(s)$:

or, $$\int_{v_0}^{v} dv = -\frac{r}{m}\int_0^s ds \text{ or } v = v_0 - \frac{rs}{m} \quad (1)$$

Thus for $$v = 0,\ s = s_{total} = \frac{m v_0}{r}$$

(c) Let we have $$\frac{m\,dv}{v} = -r v \text{ or } \frac{dv}{v} = \frac{-r}{m}dt$$

or, $$\int_0^{v_0/\eta} \frac{dv}{v} = \frac{-r}{m}\int_0^t dt, \text{ or, } \ln\frac{v_0}{\eta v_0} = -\frac{r}{m}t$$

So $$t = \frac{-m\ln(1/\eta)}{r} = \frac{m\ln\eta}{r}$$

Now, average velocity over this time interval,

$$\langle v \rangle = \frac{\int v\,dt}{\int dt} = \frac{\int_0^{\frac{m}{r}\ln\eta} v_0 e^{-\frac{rt}{m}}\,dt}{\frac{m}{r}\ln\eta} = \frac{v_0(\eta - 1)}{\eta\ln\eta}$$

1.101 According to the problem

$$m\frac{dv}{dt} = -kv^2 \text{ or, } m\frac{dv}{v^2} = -k\,dt$$

Integrating, withing the limits,

$$\int_{v_0}^{v}\frac{dv}{v^2} = -\frac{k}{m}\int_0^t dt \text{ or, } t = \frac{m}{k}\frac{(v_0 - v)}{v_0 v} \quad (1)$$

To finւ the valuė of k, rewrite

$$mv\frac{dv}{ds} = -kv^2 \text{ or, } \frac{dv}{v} = -\frac{k}{m}ds$$

On integrating

$$\int_{v_0}^{v}\frac{dv}{v} = -\frac{k}{m}\int_0^h ds$$

So, $$k = \frac{m}{h}\ln\frac{v_0}{v} \quad (2)$$

Putting the value of k from (2) in (1), we get

$$t = \frac{h(v_0 - v)}{v_0 v\ln\frac{v_0}{v}}$$

1.102 From Newton's second law for the bar in projection from, $F_x = m w_x$ along x direction we get

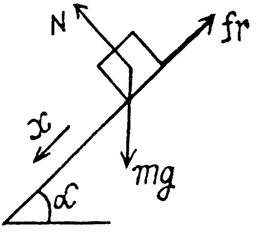
$$mg \sin\alpha - kmg\cos\alpha = mw$$

or, $$v\frac{dv}{dx} = g\sin\alpha - ax\,g\cos\alpha, \text{ (as } k = ax),$$

or, $$v\,dv = (g\sin\alpha - ax\,g\cos\alpha)\,dx$$

or, $$\int_0^v v\,dv = g\int_0^x (\sin\alpha - x\cos\alpha)\,dx$$

So, $$\frac{v^2}{2} = g\left(\sin\alpha\, x - \frac{x^2}{2}a\cos\alpha\right) \qquad (1)$$



From (1) $v = 0$ at either

$$x = 0, \text{ or } x = \frac{2}{a}\tan\alpha$$

As the motion of the bar is unidirectional it stops after going through a distance of $\frac{2}{a}\tan\alpha$.

From (1), for v_{max},

$$\frac{d}{dx}\left(\sin\alpha x - \frac{x^2}{2}a\cos\alpha\right) = 0, \text{ which yields } x = \frac{1}{a}\tan\alpha$$

Hence, the maximum velocity will be at the distance, $x = \tan\alpha/a$

Putting this value of x in (1) the maximum velocity,

$$v_{max} = \sqrt{\frac{g\sin\alpha\tan\alpha}{a}}$$

1.103 Since, the applied force is proportional to the time and the frictional force also exists, the motion does not start just after applying the force. The body starts its motion when F equals the limiting friction.

Let the motion start after time t_0, then

$$F = at_0 = kmg \text{ or, } t_0 = \frac{kmg}{a}$$

So, for $t = \leq t_0$, the body remains at rest and for $t > t_0$ obviously

$$\frac{mdv}{dt} = a(t - t_0) \quad \text{or,} \quad m\,dv = a(t - t_0)\,dt$$

Integrating, and noting $v = 0$ at $t = t_0$, we have for $t > t_0$

$$\int_0^v m\,dv = a\int_{t_0}^t (t - t_0)\,dt \quad \text{or} \quad v = \frac{a}{2m}(t - t_0)^2$$

Thus $$s = \int v\,dt = \frac{a}{2m}\int_{t_0}^t (t - t_0)^2\,dt = \frac{a}{6m}(t - t_0)^3$$

1.104 While going upward, from Newton's second law in vertical direction :

$$m\frac{v\,dv}{ds} = -(mg + kv^2) \text{ or } \frac{v\,dv}{\left(g + \frac{kv^2}{m}\right)} = -ds$$

At the maximum height h, the speed $v = 0$, so

$$\int_{v_0}^{0} \frac{v\,dv}{g + (kv^2/m)} = -\int_{0}^{h} ds$$

Integrating and solving, we get,

$$h = \frac{m}{2k}\ln\left(1 + \frac{kv_o^2}{mg}\right) \qquad (1)$$

When the body falls downward, the net force acting on the body in downward direction equals $(mg - kv^2)$,

Hence net acceleration, in downward direction, according to second law of motion

$$\frac{v\,dv}{ds} = g - \frac{kv^2}{m} \text{ or, } \frac{v\,dv}{g - \frac{kv^2}{m}} = ds$$

Thus
$$\int_{0}^{v'} \frac{v\,dv}{g - kv^2/m} = \int_{0}^{h} ds$$

Integrating and putting the value of h from (1), we get,

$$v' = v_0 / \sqrt{1 + k v_0^2/mg}.$$

1.105 Let us fix $x - y$ co-ordinate system to the given plane, taking x-axis in the direction along which the force vector was oriented at the moment $t = 0$, then the fundamental equation of dynamics expressed via the projection on x and y-axes gives,

$$F\cos\omega t = m\frac{dv_x}{dt} \qquad (1)$$

and
$$F\sin\omega t = m\frac{dv_y}{dt} \qquad (2)$$

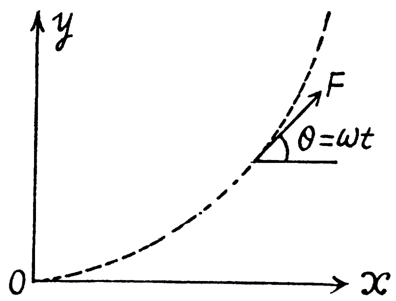
(a) Using the condition $v(0) = 0$, we obtain $v_x = \frac{F}{m\omega}\sin\omega t$ (3)

and

$$v_y = \frac{F}{m\omega}(1 - \cos\omega t) \qquad (4)$$

Hence,
$$v = \sqrt{v_x^2 + v_y^2} = \left(\frac{2F}{m\omega}\right)\left|\sin\left(\frac{\omega t}{2}\right)\right|$$

(b) It is seen from this that the velocity v turns into zero after the time interval Δt, which can be found from the relation, $\omega \frac{\Delta t}{2} = \pi$. Consequentely,



the sought distance, is

$$s = \int_0^{\Delta t} v\,dt = \frac{8F}{m\omega^2}$$

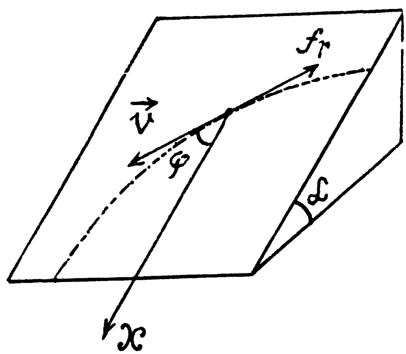
Average velocity, $< v > = \frac{\int v\,dt}{\int dt}$

So, $$<v> = \int_0^{2\pi/\omega} \frac{2F}{m\omega} \sin\left(\frac{\omega t}{2}\right) dt / (2\pi\omega) = \frac{4F}{\pi m \omega}$$

1.106 The acceleration of the disc along the plane is determined by the projection of the force of gravity on this plane $F_x = mg \sin\alpha$ and the friction force $fr = kmg \cos\alpha$. In our case $k = \tan\alpha$ and therefore

$$fr = F_x = mg \sin\alpha$$

Let us find the projection of the acceleration on the derection of the tangent to the trajectory and on the *x-axis* :



$$m\,w_t = F_x \cos\varphi - fr = mg \sin\alpha\,(\cos\varphi - 1)$$

$$m\,w_x = F_x - fr \cos\varphi = mg \sin\alpha\,(1 - \cos\varphi)$$

It is seen fromthis that $w_t = -w_x$, which means that the velocity v and its projection v_x differ only by a constant value C which does not change with time, i.e.

$$v = v_x + C,$$

where $v_x = v\cos\varphi$. The constant C is found from the initial condition $v = v_0$, whence $C = v_0$ since $\varphi = \frac{\pi}{2}$ initially. Finally we obtain

$$v = v_0 / (1 + \cos\varphi).$$

In the cource of time $\varphi \to 0$ and $v \to v_0/2$. (Motion then is unaccelerated.)

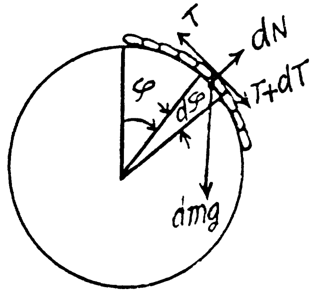
1.107 Let us consider an element of length ds at an angle φ from the vertical diameter. As the speed of this element is zero at initial instant of time, it's centripetal acceleration is zero, and hence, $dN - \lambda\,ds \cos\varphi = 0$, where λ is the linear mass density of the chain Let T and $T + dT$ be the tension at the upper and the lower ends of ds. we have from, $F_t = m\,w_t$

$$(T + dT) + \lambda\,ds\,g \sin\varphi - T = \lambda ds\,w_t$$

or, $$dT + \lambda R d\varphi\,g \sin\varphi = \lambda ds\,w_t$$

If we sum the above equation for all elements, the term $\int dT = 0$ because there is no tension at the free ends, so

$$\lambda g R \int_0^{l/R} \sin\varphi \, d\varphi = \lambda w_t \int ds = \lambda l w_t$$



Hence $\quad w_t = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$

As $w_n = a$ at initial moment

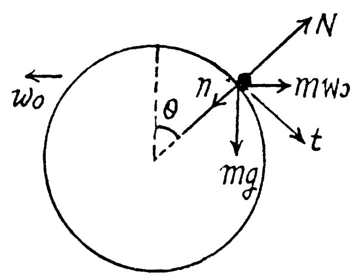
So, $w = |w_t| = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$

1.108 In the problem, we require the velocity of the body, realtive to the sphere, which itself moves with an acceleration w_0 in horizontal direction (say towards left). Hence it is advisible to solve the problem in the frame of sphere (non-inertial frame).

At an arbitary moment, when the body is at an angle θ with the vertical, we sketch the force diagram for the body and write the second law of motion in projection form $F_n = m w_n$

or, $mg\cos\theta - N - m w_0 \sin\theta = \frac{mv^2}{R} \qquad (1)$

At the break off point, $N = 0$, $\theta = \theta_0$ and let $v = v_0$, so the Eq. (1) becomes,



$\frac{v_0^2}{R} = g\cos\theta_0 - w_0\sin\theta_0 \qquad (2)$

From, $F_t = m w_t$

$mg\sin\theta - m w_0\cos\theta = m\frac{v\,dv}{ds} = m\frac{v\,dv}{R\,d\theta}$

or, $v\,dv = R\,(g\sin\theta + w_0\cos\theta)\,d\theta$

Integrating, $\int_0^{v_0} v\,dv = \int_0^{\theta_0} R\,(g\sin\theta + w_0\cos\theta)\,d\theta$

$$\frac{v_0^2}{2R} = g(1 - \cos\theta_0) + w_0\sin\theta_0 \qquad (3)$$

Note that the Eq. (3) can also be obtained by the work-energy theorem $A = \Delta T$ (in the frame of sphere)

therefore, $\qquad mgR\,(1 - \cos\theta_0) + m w_0 R\sin\theta_0 = \frac{1}{2} m v_0^2$

[here $m w_0 R\sin\theta_0$ is the work done by the pseudoforce $(-m\vec{w_o})$]

or, $\qquad \frac{v_0^2}{2R} = g\,(1 - \cos\theta_0) + w_0\sin\theta_0$

Solving Eqs. (2) and (3) we get,

$$v_0 = \sqrt{2gR/3} \text{ and } \theta_0 = \cos^{-1}\left[\frac{2 + k\sqrt{5+9k^2}}{3(1+k^2)}\right], \text{ where } k = \frac{w_0}{g}$$

Hence $$\theta_0 \Big|_{w_0 = g} = 17°$$

1.109 This is not central force problem unless the path is a circle about the said point. Rather here F_t (tangential force) vanishes. Thus equation of motion becomes,

$$v_t = v_0 = \text{constant}$$

and, $$\frac{mv_0^2}{r} = \frac{A}{r^2} \text{ for } r = r_0$$

We can consider the latter equation as the equilibrium under two forces. When the motion is perturbed, we write $r = r_0 + x$ and the net force acting on the particle is,

$$-\frac{A}{(r_0+x)^n} + \frac{mv_0^2}{r_0+x} = \frac{-A}{r_0^n}\left(1 - \frac{nx}{r_0}\right) + \frac{m\,v_0^2}{r_0}\left(1 - \frac{x}{r_0}\right) = -\frac{m\,v_0^2}{r_0^2}(1-n)\,x$$

This is opposite to the displacement x, if $n < 1$. $\left(\frac{mv_0^2}{r}\right.$ is an outward directed centrifugul force while $\frac{-A}{r^n}$ is the inward directed external force).

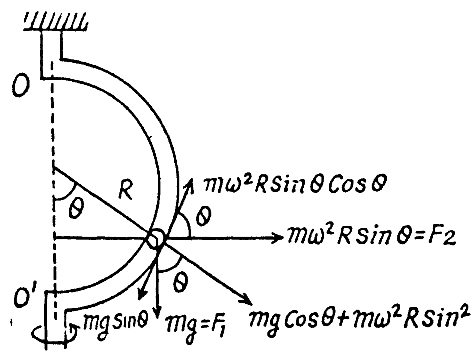
1.110 There are two forces on the sleeve, the weight F_1 and the centrifugal force F_2. We resolve both forces into tangential and normal component then the net downward tangential force on the sleeve is,

$$mg\sin\theta\left(1 - \frac{\omega^2 R}{g}\cos\theta\right)$$

This vanishes for $\theta = 0$ and for

$\theta = \theta_0 = \cos^{-1}\left(\frac{g}{\omega^2 R}\right)$, which is real if

$\omega^2 R > g$. If $\omega^2 R < g$, then $1 - \frac{\omega^2 R}{g}\cos\theta$ is always positive for small values of θ and hence the net tangential force near $\theta = 0$ opposes any displacement away from it. $\theta = 0$ is then stable.



If $\omega^2 R > g$, $1 - \frac{\omega^2 R}{g}\cos\theta$ is negative for small θ near $\theta = 0$ and $\theta = 0$ is then unstable.

However $\theta = \theta_0$ is stable because the force tends to bring the sleeve near the equilibrium position $\theta = \theta_0$.

If $\omega^2 R = g$, the two positions coincide and becomes a stable equilibrium point.

1.111 Define the axes as shown with z along the local vertical, x due east and y due north. (We assume we are in the northern hemisphere). Then the Coriolis force has the components.

$$\vec{F}_{cor} = -2m(\vec{\omega} \times \vec{v})$$

$= 2m\omega \left[v_y \cos\theta - v_z \sin\theta) \vec{i} - v_x \cos\theta \vec{j} + v_x \cos\theta \vec{k} \right] = 2m\omega (v_y \cos\theta - v_z \sin\theta) \vec{i}$

since v_x is small when the direction in which the gun is fired is due north. Thus the equation of motion (neglecting centrifugal forces) are

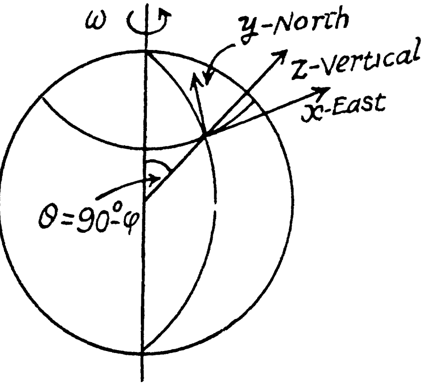
$\ddot{x} = 2m\omega (v_y \sin\varphi - v_z \cos\varphi), \ddot{y} = 0$ and $\ddot{z} = -g$

Integrating we get $\dot{y} = v$ (constant), $\dot{z} = -gt$

and $\dot{x} = 2\omega v \sin\varphi\, t + \omega g t^2 \cos\varphi$

Finally,

$x = \omega v t^2 \sin\varphi + \frac{1}{3} \omega g t^3 \cos\varphi$



Now $v >> gt$ in the present case. so,

$$x \approx \omega v \sin\varphi \left(\frac{s}{v}\right)^2 = \omega \sin\varphi \frac{s^2}{v}$$

$$\approx 7 \text{ cm (to the east).}$$

1.112 The disc exerts three forces which are mutually perpendicular. They are the reaction of the weight, mg, vertically upward, the Coriolis force $2mv'\omega$ perpendicular to the plane of the vertical and along the diameter, and $m\omega^2 r$ outward along the diameter. The resultant force is,

$$F = m\sqrt{g^2 + \omega^4 r^2 + (2v'\omega)^2}$$

1.113 The sleeve is free to slide along the rod AB. Thus only the centrifugal force acts on it. The equation is,

$$m\dot{v} = m\omega^2 r \quad \text{where } v = \frac{dr}{dt}.$$

$$\text{But } \dot{v} = v\frac{dv}{dr} = \frac{d}{dr}\left(\frac{1}{2}v^2\right)$$

so,
$$\frac{1}{2}v^2 = \frac{1}{2}\omega^2 r^2 + \text{constant}$$

or,
$$v^2 = v_0^2 + \omega^2 r^2$$

v_0 being the initial velocity when $r = 0$. The Coriolis force is then,

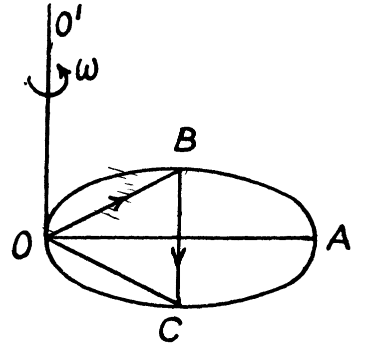
$$2m\omega\sqrt{v_0^2 + \omega^2 r^2} = 2m\omega^2 r\sqrt{1 + v_0^2/\omega^2 r^2}$$

$$= 2{\cdot}83 \text{ N on putting the values.}$$

1.114 The disc $OBAC$ is rotating with angular velocity ω about the axis OO' passing through the edge point O. The equation of motion in rotating frame is,

$$m\vec{w}' = \vec{F} + m\omega^2\vec{R} + 2m\vec{v}' \times \vec{\omega} = \vec{F} + \vec{F}_{in}$$

where $\vec{F}_{in}$ is the resultant inertial force (pseudo force) which is the vector sum of centrifugal and Coriolis forces.



(a) At A, F_{in} vanishes. Thus $0 = -2m\omega^2 R\,\hat{n} + 2mv'\,\omega\,\hat{n}$

where $\hat{n}$ is the inward drawn unit vector to the centre from the point in question, here A.

Thus, $$v' = \omega R$$

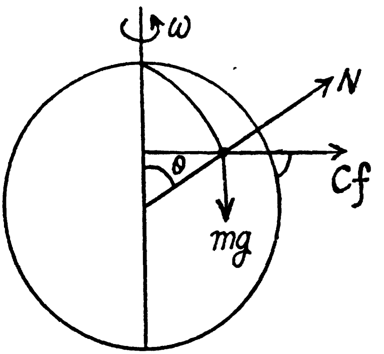
so, $$w = \frac{v'^2}{\rho} = \frac{v'^2}{R} = \omega^2 R.$$

(b) At B $$\vec{F}_{in} = m\omega^2\,\vec{OC} + m\omega^2\,\vec{BC}$$

its magnitude is $m\omega^2\sqrt{4R^2 - r^2}$, where $r = OB$.

1.115 The equation of motion in the rotating coordinate system is,

$$m\vec{w}' = \vec{F} + m\omega^2\vec{R} + 2m(\vec{v}' \times \vec{\omega})$$



Now, $\vec{v}' = R\,\dot\theta\,\vec{e}_\theta + R\sin\theta\,\dot\varphi\,\vec{e}_\varphi$

and $\vec{w}' = w'\cos\theta\,\vec{e}_r - w'\sin\theta\,\vec{e}_\theta$

$$\frac{1}{2m}\vec{F}_{cor} = \begin{vmatrix} \vec{e}_r & \vec{e}_\theta & \vec{e}_\varphi \\ 0 & R\dot\theta & R\sin\theta\,\dot\varphi \\ \omega\cos\theta & -\omega\sin\theta & 0 \end{vmatrix}$$

$$= \vec{e}_r(\omega R\sin^2\theta\,\dot\varphi) + \omega R\sin\theta\cos\theta\,\dot\varphi\,\vec{e}_\theta - \omega R\,\dot\theta\cos\theta\,\vec{e}_\theta$$

Now on the sphere,

$$\vec{v} = (-R\,\dot\theta^2 - R\sin^2\theta\,\dot\varphi^2)\,\vec{e}_r$$
$$+ (R\,\ddot\theta - R\sin\theta\cos\theta\,\dot\varphi^2)\,\vec{e}_\theta$$
$$+ (R\sin\theta\,\ddot\varphi + 2R\cos\theta\,\dot\theta\,\dot\varphi)\,\vec{e}_\varphi$$

Thus the equation of motion are,

$$m(-R\,\dot\theta^2 - R\sin^2\theta\,\dot\varphi^2) = N - mg\cos\theta + m\omega^2 R\sin^2\theta + 2m\omega R\sin^2\theta\,\dot\varphi$$

$$m(R\,\ddot\theta - R\sin\theta\cos\theta\,\dot\varphi^2) = mg\sin\theta + m\omega^2 R\sin\theta\cos\theta + 2m\omega R\sin\theta\cos\theta\,\dot\varphi$$

$$m(R\sin\theta\,\ddot\varphi + 2R\cos\theta\,\dot\theta\,\dot\varphi) = -2m\omega R\,\dot\theta\cos\theta$$

From the third equation, we get, $\dot\varphi = -\omega$

A result that is easy to understant by considering the motion in non-rotating frame. The eliminating $\dot\varphi$ we get,

$$mR\,\dot\theta^2 = mg\cos\theta - N$$
$$mR\,\ddot\theta = mg\sin\theta$$

Integrating the last equation,

$$\frac{1}{2}mR\,\dot\theta^2 = mg(1 - \cos\theta)$$

Hence $N = (3 - 2\cos\theta)\, mg$

So the body must fly off for $\theta = \theta_0 = \cos^{-1}\frac{2}{3}$, exactly as if the sphere were nonrotating.

Now, at this point F_{cf} = centrifugal force $= m\omega^2 R \sin\theta_0 = \sqrt{\frac{5}{9}}\, m\omega^2 R$

$$F_{cor} = \sqrt{\omega^2 R^2 \dot\theta^2 \cos^2\theta + (\omega^2 R^2)^2 \sin^2\theta} \times 2m$$

$$= \sqrt{\frac{5}{9}(\omega^2 R)^2 + \omega^2 R^2 \times \frac{4}{9} \times \frac{2g}{3R}} \times 2m = \frac{2}{3} m\omega^2 R \sqrt{5 + \frac{8g}{3\,\omega^2 R}}$$

1.116 (a) When the train is moving along a meridian only the Coriolis force has a lateral component and its magnitude (see the previous problem) is,

$$2m\,\omega\, v\cos\theta = 2m\,\omega \sin\lambda$$

(Here we have put $R\,\dot\theta \to v$)

So, $$F_{lateral} = 2 \times 2000 \times 10^3 \times \frac{2\pi}{86400} \times \frac{54000}{3600} \times \frac{\sqrt{3}}{2}$$

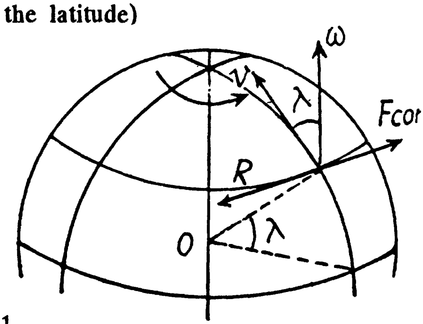
$$= 3{\cdot}77 \text{ kN, (we write } \lambda \text{ for the latitude)}$$

(b) The resultant of the inertial forces acting on the train is,

$$\vec{F}_{in} = -2m\omega R\,\dot\theta \cos\theta\, \vec{e}_\varphi$$

$$+ (m\omega^2 R \sin\theta\cos\theta + 2m\,\omega R\sin\theta\cos\theta\,\dot\varphi)\,\vec{e}_\theta$$

$$+ (m\omega^2 R\sin^2\theta + 2m\,\omega R\sin^2\theta\,\dot\varphi)\,\vec{e}_r$$



This vanishes if $\dot\theta = 0$, $\dot\varphi = -\frac{1}{2}\omega$

Thus $$\vec{v} = v_\varphi \vec{e}_\varphi,\; v_\varphi = -\frac{1}{2}\omega R\sin\theta = -\frac{1}{2}\omega R\cos\lambda$$

(We write λ for the latitude here)

Thus the train must move from the east to west along the 60th parallel with a speed,

$$\frac{1}{2}\omega R\cos\lambda = \frac{1}{4} \times \frac{2\pi}{8{\cdot}64} \times 10^{-4} \times 6{\cdot}37 \times 10^6 = 115{\cdot}8 \text{ m/s} \approx 417 \text{ km/hr}$$

1.117 We go to the equation given in 1.111. Here $v_y = 0$ so we can take $y = 0$, thus we get for the motion in the $x\,z$ plane.

$$\ddot{x} = -2\omega\, v_z \cos\theta$$

and $$\ddot{z} = -g$$

Integrating, $$z = -\frac{1}{2} g t^2$$

$$\dot{x} = \omega\, g\cos\varphi t^2$$

So $$x = \frac{1}{3}\omega\, g\cos\varphi\, t^3 = \frac{1}{3}\omega\, g\cos\varphi \left(\frac{2h}{g}\right)^{3/2}$$

$$= \frac{2\,\omega h}{3}\cos\varphi\sqrt{\frac{2h}{g}}$$

There is thus a displacement to the east of

$$\frac{2}{3} \times \frac{2\pi}{8} 64 \times 500 \times 1 \times \sqrt{\frac{400}{9{\cdot}8}} \approx 26 \text{ cm.}$$

1.3 Laws of Conservation of Energy, Momentum and Angular Momentum.

1.118 As $\vec{F}$ is constant so the sought work done

$$A = \vec{F} \cdot \Delta\vec{r} = \vec{F} \cdot (\vec{r}_2 - \vec{r}_1)$$

or, $A = (3\vec{i} + 4\vec{j}) \cdot [(2\vec{i} - 3\vec{j}) - (\vec{i} + 2\vec{j})] = (3\vec{i} + 4\vec{j}) \cdot (\vec{i} - 5\vec{j}) = 17\text{ J}$

1.119 Differentating $v(s)$ with respect to time

$$\frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}a\sqrt{s} = \frac{a^2}{2} = w$$

(As locomotive is in unidrectional motion)

Hence force acting on the locomotive $F = m\,w = \dfrac{ma^2}{2}$

Let, at $v = 0$ at $t = 0$ then the distance covered during the first t seconds

$$s = \frac{1}{2}wt^2 = \frac{1}{2}\frac{a^2}{2}t^2 = \frac{a^2}{4}t^2$$

Hence the sought work, $A = Fs = \dfrac{ma^2}{2}\dfrac{(a^2t^2)}{4} = \dfrac{m\,a^4t^2}{8}$

1.120 We have

$$T = \frac{1}{2}mv^2 = as^2 \quad \text{or,} \quad v^2 = \frac{2\,as^2}{m} \tag{1}$$

Differentating Eq. (1) with respect to time

$$2\,vw_t = \frac{4\,as}{m}v \quad \text{or,} \quad w_t = \frac{2\,as}{m} \tag{2}$$

Hence net acceleration of the particle

$$w = \sqrt{w_t^2 + w_n^2} = \sqrt{\left(\frac{2\,as}{m}\right)^2 + \left(\frac{2\,as^2}{mR}\right)^2} = \frac{2\,as}{m}\sqrt{1 + (s/R)^2}$$

Hence the sought force, $F = mw = 2as\sqrt{1 + (s/R)^2}$

1.121 Let $\vec{F}$ makes an angle θ with the horizontal at any instant of time (Fig.). Newton's second law in projection form along the direction of the force, gives :

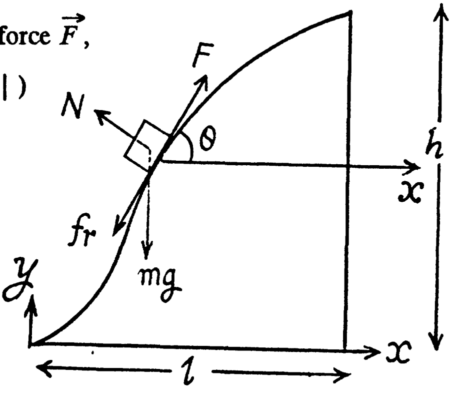
$F = kmg\cos\theta + mg\sin\theta$ (because there is no acceleration of the body.)

As $\vec{F} \uparrow\uparrow d\vec{r}$ the differential work done by the force $\vec{F}$,

$$dA = \vec{F} \cdot d\vec{r} = F\,ds, \quad (\text{where } ds = |d\vec{r}|)$$

$$= kmg\,ds\,(\cos\theta) + mg\,ds\sin\theta$$

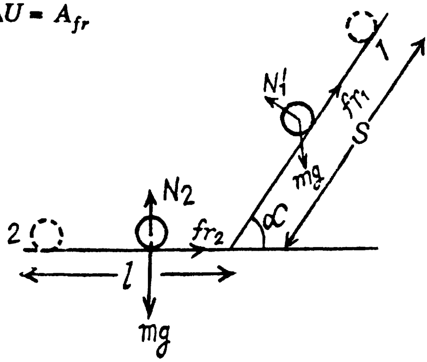
$$= kmg\,dx + mg\,dy.$$



Hence, $A = kmg\displaystyle\int_0^l dx + mg\int_0^h dy$

$$= kmg\,l + mgh = mg\,(k\,l + h).$$

1.122 Let s be the distance covered by the disc along the incline, from the Eq. of increment of M.E. of the disc in the field of gravity : $\Delta T + \Delta U = A_{fr}$



$$0 + (-mgs\sin\alpha) = -kmg\cos\alpha\, s - kmg\, l$$

or, $$s = \frac{k\,l}{\sin\alpha - k\cos\alpha} \qquad (1)$$

Hence the sought work

$$A_{fr} = -kmg\,[s\cos\alpha + l\,]$$

$$A_{fr} = -\frac{k\,lmg}{1 - k\cot\alpha} \quad \text{[Using the Eqn. (1)]}$$

On puting the values $A_{fr} = -0.05$ J

1.123 Let x be the compression in the spring when the bar m_2 is about to shift. Therefore at this moment spring force on m_2 is equal to the limiting friction between the bar m_2 and horizontal floor. Hence

$$\kappa x = k\, m_2\, g \quad \text{[where } \kappa \text{ is the spring constant (say)]} \qquad (1)$$

For the block m_1 from work-energy theorem : $A = \Delta T = 0$ for minimum force. (A here indudes the work done in stretching the spring.)

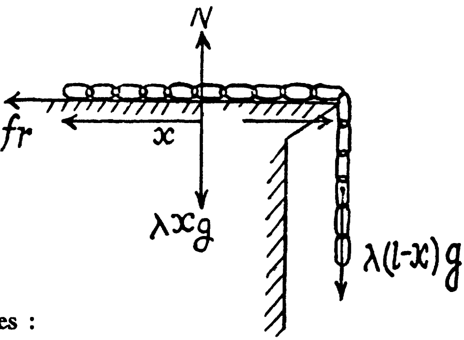
so, $$Fx - \frac{1}{2}\kappa x^2 - kmg\, x = 0 \quad \text{or} \quad \kappa\frac{x}{2} = F - km_1 g \qquad (2)$$

From (1) and (2),

$$F = kg\left(m_1 + \frac{m_2}{2}\right).$$

1.124 From the initial condition of the problem the limiting fricition between the chain lying on the horizontal table equals the weight of the over hanging part of the chain, i.e.

$\lambda\eta\, lg = k\lambda(1-\eta)\, lg$ (where λ is the linear mass density of the chain)



So, $$k = \frac{\eta}{1-\eta} \qquad (1)$$

Let (at an arbitrary moment of time) the length of the chain on the table is x. So the net friction force between the chain and the table, at this moment :

$$f_r = kN = k\lambda x g \qquad (2)$$

The differential work done by the friction forces :

$$dA = \vec{f_r}\cdot d\vec{r} = -f_r\, ds = -k\lambda x g\,(-dx) = \lambda g\left(\frac{\eta}{1-\eta}\right)x\, dx \qquad (3)$$

(Note that here we have written $ds = -dx$., because ds is essentially a positive term and as the length of the chain decreases with time, dx is negative)

Hence, the sought work done

$$A = \int_{(1-\eta)l}^{0} \lambda g \frac{\eta}{1-\eta} x\, dx = -(1-\eta)\,\eta\frac{mgl}{2} = -1{\cdot}3 \text{ J}$$

1.125 The velocity of the body, t seconds after the begining of the motion becomes $\vec{v} = \vec{v}_0 + \vec{g}t$. The power developed by the gravity ($m\vec{g}$) at that moment, is

$$P = m\vec{g} \cdot \vec{v} = m(\vec{g} \cdot \vec{v}_0 + g^2 t) = mg(gt - v_0 \sin\alpha) \quad (1)$$

As $m\vec{g}$ is a constant force, so the average power

$$<P> = \frac{A}{\tau} = \frac{m\vec{g} \cdot \Delta\vec{r}}{\tau}$$

where $\Delta\vec{r}$ is the net displacement of the body during time of flight.

As, $\quad m\vec{g} \perp \Delta\vec{r}$ so $<P> = 0$

1.126 We have $\quad w_n = \frac{v^2}{R} = at^2, \quad \text{or,} \quad v = \sqrt{aR}\, t,$

t is defined to start from the begining of motion from rest.

So, $\quad w_t = \frac{dv}{dt} = \sqrt{aR}$

Instantaneous power, $P = \vec{F} \cdot \vec{v} = m(w_t \hat{u}_t + w_n \hat{u}_t) \cdot (\sqrt{aR}\, t\, \hat{u}_t),$

(where $\hat{u}_t$ and $\hat{u}_t$ are unit vectors along the direction of tangent (velocity) and normal respectively)

So, $\quad P = m w_t \sqrt{aR}\, t = maRt$

Hence the sought average power

$$<P> = \frac{\int_0^t P\, dt}{\int_0^t dt} = \frac{\int_0^t maRt\, dt}{t}$$

Hence $\quad <P> = \frac{maRt^2}{2t} = \frac{maRt}{2}$

1.127 Let the body m acquire the horizontal velocity v_0 along positive x-axis at the point O.

(a) Velocity of the body t seconds after the begining of the motion,

$$\vec{v} = \vec{v}_0 + \vec{w}t = (v_0 - kgt)\vec{i} \quad (1)$$

Instantaneous power $P = \vec{F} \cdot \vec{v} = (-kmg\vec{i}) \cdot (v_0 - kgt)\vec{i} = -kmg(v_0 - kgt)$

From Eq. (1), the time of motion $\tau = v_0/kg$

Hence sought average power during the time of motion

$$<P> = \frac{\int_0^\tau -kmg(v_0 - kgt)\, dt}{\tau} = -\frac{kmg\, v_0}{2} = -2\text{ W (On substitution)}$$

From $F_x = m w_x$

$$-kmg = m w_x = m v_x \frac{dv_x}{dx}$$

or, $\quad v_x\, dv_x = -kg\, dx = -\alpha g x\, dx$

To find $v(x)$, let us integrate the above equation

$$\int_{v_0}^{v} v_x\, dv_x = -\alpha g \int_0^x x\, dx \quad \text{or,} \quad v^2 = v_0^2 - \alpha g x^2 \tag{1}$$

Now, $$\vec{P} = \vec{F} \cdot \vec{v} = -m\alpha x g \sqrt{v_0^2 - \alpha g x^2} \tag{2}$$

For maximum power, $\frac{d}{dt}\left(\sqrt{v_0^2 x^2 - \lambda g x^4}\right) = 0$ which yields $x = \frac{v_0}{\sqrt{2\alpha g}}$

Putting this value of x, in Eq. (2) we get,

$$P_{max} = -\frac{1}{2} m v_0^2 \sqrt{\alpha g}$$

1.128 Centrifugal force of inertia is directed outward along radial line, thus the sought work

$$A = \int_{r_1}^{r_2} m\omega^2 r\, dr = \frac{1}{2} m\omega^2 \left(r_2^2 - r_1^2\right) = 0{\cdot}20\ \text{T} \quad \text{(On substitution)}$$

1.129 Since the springs are connected in series, the combination may be treated as a single spring of spring constant.

$$\kappa = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

From the equation of increment of M.E., $\Delta T + \Delta U = A_{ext}$

$$0 + \frac{1}{2}\kappa\, \Delta l^2 = A, \quad \text{or,} \quad A = \frac{1}{2}\left(\frac{\kappa\, \kappa_2}{\kappa_1 + \kappa_2}\right)\Delta l^2$$

1.130 First, let us find the total height of ascent. At the beginning and the end of the path of velocity of the body is equal to zero, and therefore the increment of the kinetic energy of the body is also equal to zero. On the other hand, in according with work-energy theorem ΔT is equal to the algebraic sum of the works A performed by all the forces, i.e. by the force F and gravity, over this path. However, since $\Delta T = 0$ then $A = 0$. Taking into account that the upward direction is assumed to coincide with the positive direction of the y - axis, we can write

$$A = \int_0^h (\vec{F} + m\vec{g}) \cdot d\,\vec{r} = \int_0^h (F_y - mg)\, dy$$

$$= mg \int_0^h (1 - 2\,ay)\, dy = mgh\,(1 - ah) = 0.$$

whence $h = 1/a$.

The work performed by the force F over the first half of the ascent is

$$A_F = \int_0^{h/2} F_y\, dy = 2mg \int_0^{h/2} (1 - ay)\, dy = 3\,mg/4a.$$

The corresponding increment of the potential energy is

$$\Delta U = mgh/2 = mg/2a.$$

1.131 From the equation $F_r = -\frac{dU}{dr}$ we get $F_r = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$

(a) we have at $r = r_0$, the particle is in equilibrium position. i.e. $F_r = 0$ so, $r_0 = \frac{2a}{b}$

To check, whether the position is steady (the position of stable equilibrium), we have to satisfy

$$\frac{d^2U}{dr^2} > 0$$

We have

$$\frac{d^2U}{dr^2} = \left[\frac{6a}{r^4} - \frac{2b}{r^3}\right]$$

Putting the value of $r = r_0 = \frac{2a}{b}$, we get

$\frac{d^2U}{dr^2} = \frac{b^4}{8a^3}$, (as a and b are positive constant)

So,

$$\frac{d^2U}{dr^2} = \frac{b^2}{8a^3} > 0,$$

which indicates that the potential energy of the system is minimum, hence this position is steady.

(b) We have

$$F_r = -\frac{dU}{dr} = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$$

For F_r to be maximum, $\frac{dF_r}{dr} = 0$

So, $r = \frac{3a}{b}$ and then $F_{r\,(\max)} = \frac{-b^3}{27a^2}$,

As F_r is negative, the force is attractive.

1.132 (a) We have

$$F_x = -\frac{\partial U}{dx} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{dy} = -2\beta y$$

So,

$$\vec{F} = 2\alpha x\vec{i} - 2\beta y\vec{i} \text{ and, } F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} \qquad (1)$$

For a central force, $\vec{r} \times \vec{F} = 0$

Here,

$$\vec{r} \times \vec{F} = (x\vec{i} + y\vec{j}) \times (-2\alpha x\vec{i} - 2\beta y\vec{j})$$

$$= -2\beta xy\vec{k} - 2\alpha xy(\vec{k}) \neq 0$$

Hence the force is not a central force.

(b) As $U = \alpha x^2 + \beta y^2$

So,

$$F_x = \frac{\partial U}{\partial x} = -2\alpha x \text{ and } F_y = \frac{-\partial U}{\partial y} = -2\beta y.$$

So,

$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4\alpha^2 x^2 + 4\beta^2 y^2}$$

According to the problem

$$F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} = C \text{ (constant)}$$

or, $$\alpha^2 x^2 + \beta^2 y^2 = \frac{C^2}{2}$$

or, $$\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = \frac{C^2}{2\,\alpha^2\beta^2} = k\,(\text{say}) \tag{2}$$

Therefore the surfaces for which F is constant is an ellipse.

For an equipotential surface U is constant.

So, $$\alpha x^2 + \beta y^2 = C_0 \text{ (constant)}$$

or, $$\frac{x^2}{\sqrt{\beta^2}} + \frac{y^2}{\sqrt{\alpha^2}} = \frac{C_0}{\alpha\,\beta} = K_0 \text{ (constant)}$$

Hence the equipotential surface is also an ellipse.

1.133 Let us calculate the work performed by the forces of each field over the path from a certain point 1 (x_1, y_1) to another certain point 2 (x_2, y_2)

(i) $dA = \vec{F}\cdot d\vec{r} = ay\,\vec{i}\cdot d\vec{r} = aydx$ or, $A = a\int_{x_1}^{x_2} y\,dx$

(ii) $dA = \vec{F}\cdot d\vec{r} = (ax\vec{i} + by\vec{i}\,)\cdot d\vec{r} = axdx + bydy$

Hence $$A = \int_{x_1}^{x_2} a\,xdx + \int_{y_1}^{y_2} bydy$$

In the first case, the integral depends on the function of type $y(x)$, i.e. on the shape of the path. Consequently, the first field of force is not potential. In the second case, both the integrals do not depend on the shape of the path. They are defined only by the coordinate of the initial and final points of the path, therefore the second field of force is potential.

1.134 Let s be the sought distance, then from the equation of increment of M.E. $\Delta T + \Delta U = A_{fr}$

$$\left(0 - \frac{1}{2}mv_0^2\right) + (+\,mg\,s\sin\alpha) = -\,kmg\cos\alpha\,s$$

or, $$s = \frac{v_0^2}{2g}\Big/(\sin\alpha + k\cos\alpha)$$

Hence $$A_{fr} = -\,kmg\cos\alpha\,s = \frac{-\,kmv_0^2}{2\,(k + \tan\alpha)}$$

1.135 Velocity of the body at height h, $v_h = \sqrt{2g\,(H-h)}$, horizontally (from the figure given in the problem). Time taken in falling through the distance h.

$t = \sqrt{\frac{2h}{g}}$ (as initial vertical component of the velocity is zero.)

Now $$s = v_h\,t = \sqrt{2g\,(H+h)} \times \sqrt{\frac{2h}{g}} = \sqrt{4\,(Hh - h^2)}$$

For s_{max}, $\frac{d}{ds}(Hh - h^2) = 0$, which yields $h = \frac{H}{2}$

Putting this value of h in the expression obtained for s, we get,

$$s_{max} = H$$

1.136 To complete a smooth vertical track of radius R, the minimum height at which a particle starts, must be equal to $\frac{5}{2}R$ (one can proved it from energy conservation). Thus in our problem body could not reach the upper most point of the vertical track of radius $R/2$. Let the particle A leave the track at some point O with speed v (Fig.). Now from energy conservation for the body A in the field of gravity :

$$mg\left[h - \frac{h}{2}(1 + \sin\theta)\right] = \frac{1}{2}mv^2$$

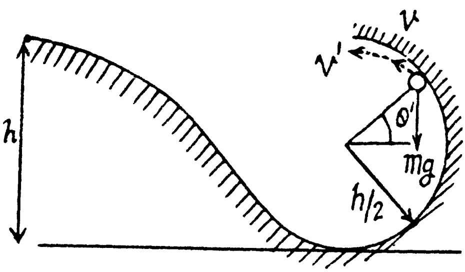
or, $\quad v^2 = gh(1 - \sin\theta) \qquad (1)$

From Newton's second law for the particle at the point O; $F_n = mw_n$,

$$N + mg\sin\theta = \frac{mv^2}{(h/2)}$$

But, at the point O the normal reaction $N = 0$

So, $\quad v^2 = \frac{gh}{2}\sin\theta \qquad (2)$



From (3) and (4), $\sin\theta = \frac{2}{3}$ and $v = \sqrt{\frac{gh}{3}}$

After leaving the track at O, the particle A comes in air and further goes up and at maximum height of it's trajectory in air, it's velocity (say v') becomes horizontal (Fig.). Hence, the sought velocity of A at this point.

$$v' = v\cos(90 - \theta) = v\sin\theta = \frac{2}{3}\sqrt{\frac{gh}{3}}$$

1.137 Let, the point of suspension be shifted with velocity v_A in the horizontal direction towards left then in the rest frame of point of suspension the ball starts with same velocity horizontally towards right. Let us work in this, frame. From Newton's second law in projection form towards the point of suspension at the upper most point (say B) :

$$mg + T = \frac{mv_B^2}{l} \quad \text{or,} \quad T = \frac{mv_B^2}{l} - mg \qquad (1)$$

Condition required, to complete the vertical circle is that $T \geq 0$. But $\qquad (2)$

$$\frac{1}{2}mv_A^2 = mg(2l) + \frac{1}{2}mv_B^2 \quad \text{So,} \quad v_B^2 = v_A^2 - 4gl \qquad (3)$$

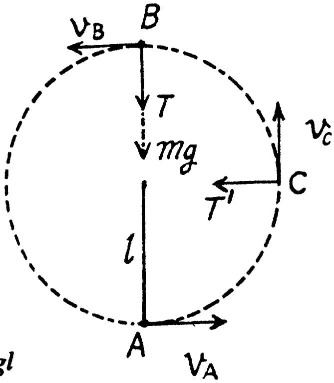
From (1), (2) and (3)

$$T = \frac{m(v_A^2 - 4gl)}{l} - mg \geq 0 \quad \text{or,} \quad v_A \geq \sqrt{5gl}$$

Thus $\quad v_{A\,(\min)} = \sqrt{5gl}$

From the equation $F_n = mw_n$ at point C

$$T' = \frac{mv_c^2}{l} \qquad (4)$$



Again from energy conservation

$$\frac{1}{2}mv_A^2 = \frac{1}{2}mv_c^2 + mgl \qquad (5)$$

From (4) and (5)

$$T = 3\,mg$$

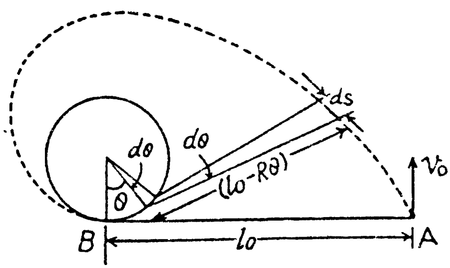
1.138 Since the tension is always perpendicular to the velocity vector, the work done by the tension force will be zero. Hence, according to the work energy theorem, the kinetic energy or velocity of the disc will remain constant during it's motion. Hence, the sought time $t = \frac{s}{v_0}$, where s is the total distance traversed by the small disc during it's motion.

Now, at an arbitary position (Fig.)

$ds = (l_0 - R\,\theta)\,d\,\theta,$

$$\text{so,} \quad s = \int_0^{l/R} (l_0 - R\,\theta)\,d\,\theta$$

$$\text{or,} \quad s = \frac{l_0^2}{R} - \frac{R\,l_0^2}{2R^2} = \frac{l_0^2}{2R}$$



Hence, the required time, $\quad t = \frac{l_0^2}{2R\,v_0}$

It should be clearly understood that the only uncompensated force acting on the disc A in this case is the tension T, of the thread. It is easy to see that there is no point here, relative to which the moment of force T is invarible in the process of motion. Hence conservation of angular momentum is not applicable here.

1.139 Suppose that Δl is the elongation of the rubbler cord. Then from energy conservation,

$$\Delta U_{gr} + \Delta U_{el} = 0 \text{ (as } \Delta T = 0)$$

$$\text{or,} \quad -mg(l + \Delta l) + \frac{1}{2}\kappa\,\Delta l^2 = 0$$

$$\text{or,} \quad \frac{1}{2}\kappa\,\Delta l^2 - mg\,\Delta l - mg\,l = 0$$

or, $$\Delta l = \frac{mg \pm \sqrt{(mg)^2 + 4 \times \frac{\kappa}{2} mgl}}{2 \times \frac{\kappa}{2}} \times \frac{\kappa}{2} = \frac{mg}{\kappa}\left[1 + \sqrt{1 \pm \frac{2\kappa l}{mg}}\right]$$

Since the value of $\sqrt{1 + \frac{2\kappa l}{mg}}$ is certainly greater than 1, hence negative sign is avoided.

So, $$\Delta l = \frac{mg}{\kappa}\left(1 + \sqrt{1 + \frac{2\kappa l}{mg}}\right)$$

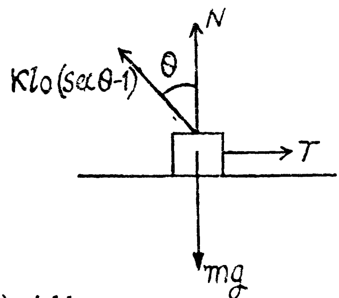
1.140 When the thread *PA* is burnt, obviously the speed of the bars will be equal at any instant of time until it breaks off. Let v be the speed of each block and θ be the angle, which the elongated spring makes with the vertical at the moment, when the bar A breaks off the plane. At this stage the elongation in the spring.

$$\Delta l = l_0 \sec\theta - l_0 = l_0(\sec\theta - 1) \quad (1)$$

Since the problem is concerned with position and there are no forces other than conservative forces, the mechanical energy of the system (both bars + spring) in the field of gravity is conserved, i.e. $\Delta T + \Delta U = 0$

So, $$2\left(\frac{1}{2}mv^2\right) + \frac{1}{2}\kappa l_0^2(\sec\theta - 1)^2 - mgl_0\tan\theta = 0 \quad (2)$$

From Newton's second law in projection form along vertical direction :



$mg = N + \kappa l_0(\sec\theta - 1)\cos\theta$

But, at the moment of break off, $N = 0$.

Hence, $\kappa l_0(\sec\theta - 1)\cos\theta = mg$

or, $$\cos\theta = \frac{\kappa l_0 - mg}{\kappa l_0} \quad (3)$$

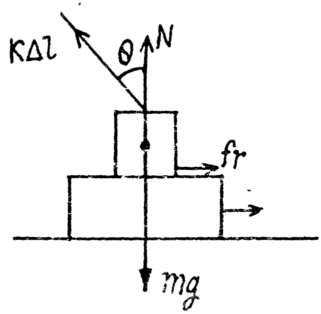
Taking $\kappa = \frac{5\,mg}{l_0}$, simultaneous solution of (2) and (3) yields :

$$v = \sqrt{\frac{19\,g\,l_0}{32}} = 1{\cdot}7\text{ m/s}.$$

1.141 Obviously the elongation in the cord, $\Delta l = l_0(\sec\theta - 1)$, at the moment the sliding first starts and at the moment horizontal projection of spring force equals the limiting friction.

So, $$\kappa_1 \Delta l \sin\theta = kN \quad (1)$$

(where κ_1 is the elastic constant).



From Newton's law in projection form along vertical direction :

$$\kappa_1 \Delta l \cos\theta + N = mg.$$

or, $$N = mg - \kappa_1 \Delta l \cos\theta \quad (2)$$

From (1) and (2),

$$\kappa_1 \Delta l \sin\theta = k(mg - \kappa_1 \Delta l \cos\theta)$$

or, $\kappa_1 = \dfrac{kmg}{\Delta l \sin\theta + k\,\Delta l \cos\theta}$

From the equation of the increment of mechanical energy : $\Delta U + \Delta T = A_{fr}$

or, $$\left(\frac{1}{2}\kappa_1\,\Delta l^2\right) = A_{fr}$$

or, $$\frac{kmg\,\Delta l^2}{2\,\Delta l\,(\sin\theta + k\cos\theta)} = A_{fr}$$

Thus $$A_{fr} = \frac{kmg\,l_0\,(\sec\theta - 1)}{2\,(\sin\theta - k\cos\theta)} = 0{\cdot}09\ \text{J} \quad \text{(on substitution)}$$

1.142 Let the deformation in the spring be Δl, when the rod AB has attained the angular velocity ω. From the second law of motion in projection form $F_n = mw_n$.

$$\kappa\,\Delta l = m\,\omega^2\,(l_0 + \Delta l) \quad \text{or,} \quad \Delta l = \frac{m\omega^2\,l_0}{\kappa - m\omega^2}$$

From the energy equation, $A_{ext} = \dfrac{1}{2}mv^2 + \dfrac{1}{2}\kappa\,\Delta l^2$

$$= \frac{1}{2}m\omega^2\,(l_0 + \Delta l)^2 + \frac{1}{2}\kappa\,\Delta l^2$$

$$= \frac{1}{2}m\omega^2\left(l_0 + \frac{m\omega^2\,l_0}{\kappa - m\omega^2}\right)^2 + \frac{1}{2}\kappa\left(\frac{m\omega^2\,l_0^2}{\kappa - m\omega^2}\right)^2$$

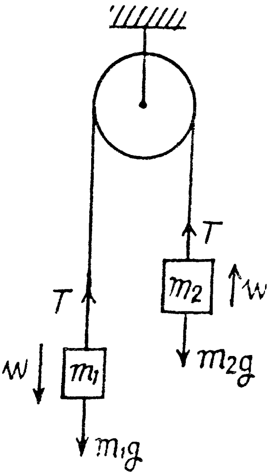
On solving $$A_{ext} = \frac{\kappa}{2}\,\frac{l_0^2\,\eta\,(1+\eta)}{(1-\eta)^2}, \quad \text{where } \eta = \frac{m\omega^2}{\kappa}$$

1.143 We know that acceleration of centre of mass of the system is given by the expression.

$$\vec{w}_C = \frac{m_1\,\vec{w}_1 + m_2\,\vec{w}_2}{m_1 + m_2}$$

Since $$\vec{w}_1 = -\vec{w}_2$$

$$\vec{w}_C = \frac{(m_1 - m_2)\,\vec{w}_1}{m_1 + m_2} \qquad (1)$$



Now from Newton's second law $\vec{F} = m\vec{w}$, for the bodies m_1 and m_2 respectively.

$$\vec{T} + m_1\,\vec{g} = m_1\,\vec{w}_1 \qquad (2)$$

and $$\vec{T} + m_2\,\vec{g} = m_2\,\vec{w}_2 = -\,m_2\,\vec{w}_1 \qquad (3)$$

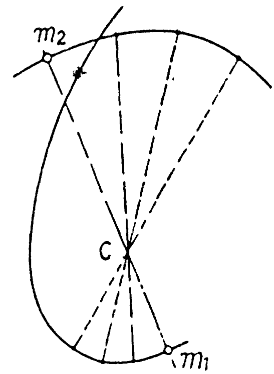
Solving (2) and (3)

$$\vec{w}_1 = \frac{(m_1 - m_2)\,\vec{g}}{m_1 + m_2} \qquad (4)$$

Thus from (1), (2) and (4),

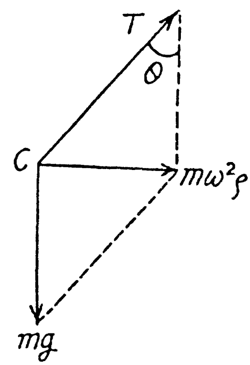
$$\vec{w}_c = \frac{(m_1 - m_2)^2 \vec{g}}{(m_1 + m_2)^2}$$

1.144 As the closed system consisting two particles m_1 and of m_2 is initially at rest the C.M. of the system will remain at rest. Further as $m_2 = m_1/2$, the C.M. of the system divides the line joining m_1 and m_2 at all the moments of time in the ratio 1 : 2. In addition to it the total linear momentum of the system at all the times is zero. So, $\vec{p_1} = -\vec{p_2}$ and therefore the velocities of m_1 and m_2 are also directed in opposite sense. Bearing in mind all these thing, the sought trajectory is as shown in the figure.



1.145 First of all, it is clear that the chain does not move in the vertical direction during the uniform rotation. This means that the vertical component of the tension T balances gravity. As for the horizontal component of the tension T, it is constant in magnitude and permanently directed toward the rotation axis. It follows from this that the C.M. of the chain, the point C, travels along horizontal circle of radius ρ (say). Therefore we have,

$$T \cos \theta = mg \quad \text{and} \quad T \sin \theta = m\omega^2 \rho$$



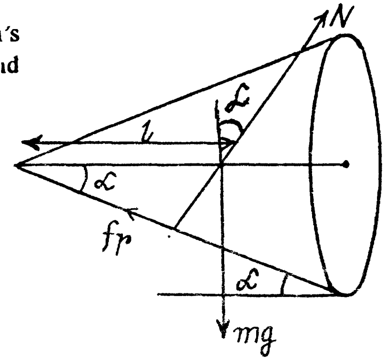
Thus $\rho = \frac{g \tan \theta}{\omega^2} = 0{\cdot}8 \text{ cm}$

and $T = \frac{mg}{\cos \theta} = 5 \text{ N}$

1.146 (a) Let us draw free body diagram and write Newton's second law in terms of projection along vertical and horizontal direction respectively.

$$N \cos \alpha - mg + fr \sin \alpha = 0 \qquad (1)$$

$$fr \cos \alpha - N \sin \alpha = m\omega^2 l \qquad (2)$$



From (1) and (2)

$$fr \cos \alpha - \frac{\sin \alpha}{\cos \alpha}(-fr \sin \alpha + mg) = m\omega^2 l$$

So, $$fr = mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) = 6\text{N} \quad (3)$$

(b) For rolling, without sliding,

$$fr \le k\text{N}$$

but, $N = mg\cos\alpha - m\,\omega^2 l\sin\alpha$

$$mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) \le k(mg\cos\alpha - m\,\omega^2 l\sin\alpha) \text{ [Using (3)]}$$

Rearranging, we get,

$$m\,\omega^2 l(\cos\alpha + k\sin\alpha) \le (k\,mg\cos\alpha - mg\sin\alpha)$$

Thus $$\omega \le \sqrt{g(k - \tan\alpha)/(1 + k\tan\alpha)\,l} = 2 \text{ rad/s}$$

1.147 (a) Total kinetic energy in frame K' is

$$T = \frac{1}{2}m_1(\vec{v}_1 - \vec{V})^2 + \frac{1}{2}m_2(\vec{v}_2 - \vec{V})^2$$

This is minimum with respect to variation in $\vec{V}$, when

$$\frac{\delta T'}{\delta\vec{V}} = 0, \text{ i.e. } m_1(\vec{v}_1 - \vec{V})^2 + m_2(\vec{v}_2 - \vec{V}) = 0$$

or $$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{V}_2}{m_1 + m_2} = \vec{v}_c$$

Hence, it is the frame of C.M. in which kinetic energy of a system is minimum.

(b) Linear momentum of the particle 1 in the K' or C frame

$$\vec{\tilde{p}}_1 = m_1(\vec{v}_1 - \vec{v}_c) = \frac{m_1 m_2}{m_1 + m_2}(\vec{v}_1 - \vec{v}_2)$$

or, $$\vec{\tilde{p}}_1 = \mu(\vec{v}_1 - \vec{v}_2), \text{ where, } \mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}$$

Similarly, $$\vec{\tilde{p}}_2 = \mu(\vec{v}_2 - \vec{v}_1)$$

So, $$|\vec{\tilde{p}}_1| = |\vec{\tilde{p}}_2| = \tilde{p} = \mu\, v_{rel} \text{ where, } v_{rel} = |\vec{v}_1 - \vec{v}_2| \quad (3)$$

Now the total kinetic energy of the system in the C frame is

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 = \frac{\tilde{p}^2}{2m_1} + \frac{\tilde{p}^2}{2m_2} = \frac{\tilde{p}^2}{2\mu}$$

Hence $$\tilde{T} = \frac{1}{2}\mu\, v_{rel}^2 = \frac{1}{2}\mu\,|\vec{v}_1 - \vec{v}_2|^2$$

1.148 To find the relationship between the values of the mechanical energy of a system in the K and C reference frames, let us begin with the kinetic energy T of the system. The velocity of the i-th particle in the K frame may be represented as $\vec{v}_i = \vec{\tilde{v}}_i + \vec{v}_C$. Now we can write

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\vec{\tilde{v}}_1 + \vec{v}_C) \cdot (\vec{\tilde{v}}_i + \vec{v}_C)$$

$$= \sum \frac{1}{2} m_i \tilde{v}_i^2 + \vec{v}_C \sum m_i \vec{\tilde{v}}_1 + \sum \frac{1}{2} m_i v_C^2$$

Since in the C frame $\sum m_i \vec{\tilde{v}}_i = 0$, the previous expression takes the form

$$T = \tilde{T} + \frac{1}{2} m\, v_C^2 = \tilde{T} + \frac{1}{2} m\, V^2 \text{ (since according to the problem } v_C = V) \quad (1)$$

Since the internal potential energy U of a system depends only on its configuration, the magnitude U is the same in all refrence frames. Adding U to the left and right hand sides of Eq. (1), we obtain the sought relationship

$$E = \tilde{E} + \frac{1}{2} m\, V^2$$

1.149 As initially $U = \tilde{U} = 0$, so, $\tilde{E} = \tilde{T}$

From the solution of 1.147 (b)

$$\tilde{T} = \frac{1}{2} \mu \,|\vec{v}_1 - \vec{v}_2|,$$

As

$$\vec{v}_1 \perp \vec{v}_2$$

Thus

$$\tilde{T} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1^2 + v_2^2)$$

1.150 Velocity of masses m_1 and m_2, after t seconds are respectively.

$$\vec{v}_1{}' = \vec{v}_1 + \vec{g}\,t \text{ and } \vec{v}_2{}' = \vec{v}_2 + \vec{g}\,t$$

Hence the final momentum of the system,

$$\vec{p} = m_1 \vec{v}_1{}' + m_2 \vec{v}_2{}' = m_1 \vec{v}_1 + m_2 \vec{v}_2 + (m_1 + m_2)\, \vec{g}\,t$$

$$= \vec{p}_0 + m\, \vec{g}\,t, \text{ (where, } \vec{p}_0 = m_2 \vec{v}_1 + m_2 \vec{v}_2 \text{ and } m = m_1 + m_2)$$

And radius vector,

$$\vec{r}_C = \vec{v}_C\, t + \frac{1}{2} \vec{w}_C\, t^2$$

$$\frac{(m_1 \vec{v}_1 + m_2 \vec{v}_2)\, t}{(m_1 + m_2)} + \frac{1}{2} \vec{g}\, t^2$$

$$= \vec{v}_0\, t + \frac{1}{2} \vec{g}\, t^2, \text{ where } \vec{v}_0 = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

1.151 After releasing the bar 2 acquires the velocity v_2, obtained by the energy, conservation :

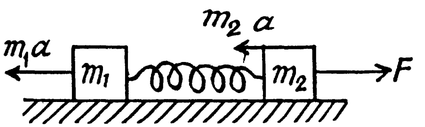
$$\frac{1}{2}m_2 v_2^2 = \frac{1}{2}\kappa x^2 \text{ or, } v_2 = x\sqrt{\frac{\kappa}{m_2}} \qquad (1)$$

Thus the sought velocity of C.M.

$$v_{cm} = \frac{0 + m_2 x\sqrt{\frac{\kappa}{m_2}}}{m_1 + m_2} = \frac{x\sqrt{m_2 \kappa}}{(m_1 + m_2)}$$

1.152 Let us consider both blocks and spring as the physical system. The centre of mass of the system moves with acceleration $a = \frac{F}{m_1 + m_2}$ towards right. Let us work in the frame of centre of mass. As this frame is a non-inertial frame (accelerated with respect to the ground) we have to apply a pseudo force $m_1 a$ towards left on the block m_1 and $m_2 a$ towards left on the block m_2

As the center of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The elongation of the spring will be maximum or minimum at this instant. Assume that the block m_1 is displaced by the distance x_1 and the block m_2 through a distance x_2 from the initial positions.



From the energy equation in the frame of C.M.

$$\Delta\tilde{T} + U = A_{ext},$$

(where A_{ext} also includes the work done by the pseudo forces)

Here,

$\Delta\tilde{T} = 0$, $U = \frac{1}{2}k(x_1 + x_2)^2$ and

$$W_{ext} = \left(\frac{F - m_2 F}{m_1 + m_2}\right)x_2 + \frac{m_1 F}{m_1 + m_2}x_1 = \frac{m_1 F(x_1 + x_2)}{m_1 + m_2},$$

or, $$\frac{1}{2}k(x_1 + x_2)^2 = \frac{m_1(x_1 + x_2)F}{m_1 + m_2}$$

So, $$x_1 + x_2 = 0 \text{ or, } x_1 + x_2 = \frac{2m_1 F}{k(m_1 + m_2)}$$

Hence the maximum separation between the blocks equals : $l_0 + \frac{2m_1 F}{k(m_1 + m_2)}$

Obviously the minimum sepation corresponds to zero elongation and is equal to l_0

1.153 (a) The initial compression in the spring Δl must be such that after burning of the thread, the upper cube rises to a height that produces a tension in the spring that is atleast equal to the weight of the lower cube. Actually, the spring will first go from its compressed

state to its natural length and then get elongated beyond this natural length. Let l be the maximum elongation produced under these circumstances.
Then

$$\kappa l = mg \tag{1}$$

Now, from energy conservation,

$$\frac{1}{2}\kappa \Delta l^2 = mg(\Delta l + l) + \frac{1}{2}\kappa l^2 \tag{2}$$

(Because at maximum elongation of the spring, the speed of upper cube becomes zero)
From (1) and (2),

$$\Delta l^2 - \frac{2mg\,\Delta l}{\kappa} - \frac{3\,m^2 g^2}{\kappa^2} = 0 \text{ or, } \Delta l = \frac{3mg}{\kappa}, \frac{-mg}{\kappa}$$

Therefore, acceptable solution of Δl equals $\frac{3mg}{\kappa}$

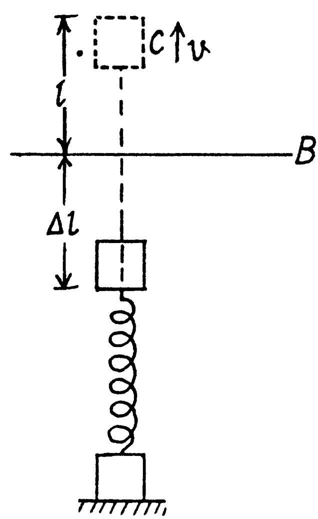
(b) Let v the velocity of upper cube at the position (say, at C) when the lower block breaks off the floor, then from energy conservation.

$$\frac{1}{2}mv^2 = \frac{1}{2}\kappa(\Delta l^2 - l^2) - mg(l + \Delta l)$$

$$(\text{where } l = mg/\kappa \text{ and } \Delta l = 7\frac{mg}{\kappa})$$

or,
$$v^2 = 32\frac{mg^2}{\kappa} \tag{2}$$

At the position C, the velocity of C.M; $v_C = \frac{mv + 0}{2m} = \frac{v}{2}$ –Let, the C.M. of the system (spring+ two cubes) further rises up to Δy_{C2}.



Now, from energy conservation,

$$\frac{1}{2}(2m)\,v_C^2 = (2m)\,g\,\Delta r_{C2}$$

or, $$\Delta y_{C2} = \frac{v_C^2}{2g} = \frac{v^2}{8g} = \frac{4\,mg}{\kappa}$$

But, uptil position C, the C.M. of the system has already elevated by,

$$\Delta y_{C1} = \frac{(\Delta l + l)\,m + 0}{2m} = \frac{4\,mg}{\kappa}$$

Hence, the net displacement of the C.M. of the system, in upward direction

$$\Delta y_C = \Delta y_{C1} + \Delta y_{C2} = \frac{8\,mg}{\kappa}$$

1.154 Due to ejection of mass from a moving system (which moves due to inertia) in a direction perpendicular to it, the velocity of moving system does not change. The momentum change being adjusted by the forces on the rails. Hence in our problem velocities of buggies change only due to the entrance of the man coming from the other buggy. From the

Solving (1) and (2), we get

$$v_1 = \frac{mv}{M-m} \text{ and } v_2 = \frac{Mv}{M-m}$$

As $\vec{v_1} \uparrow\downarrow \vec{v}$ and $\vec{v_2} \uparrow\uparrow \vec{v}$

So, $$\vec{v_1} = \frac{-m\vec{v}}{(M-m)} \text{ and } \vec{v_2} = \frac{M\vec{v}}{(M-m)}$$

1.155 From momentum conservation, for the system "rear buggy with man"

$$(M+m)\vec{v_0} = m(\vec{u} + \vec{v_R}) + M\vec{v_R} \qquad (1)$$

From momentum conservation, for the system (front buggy + man coming from rear buggy)

$$M\vec{v_0} + m(\vec{u} + \vec{v_R}) = (M+m)\vec{v_F}$$

So, $$\vec{v_F} = \frac{M\vec{v_0}}{M+m} + \frac{m}{M+m}(\vec{u} + \vec{v_R})$$

Putting the value of $\vec{v_R}$ from (1), we get

$$\vec{v_F} = \vec{v_0} + \frac{mM}{(M+m)^2}\vec{u}$$

1.156 (i) Let $\vec{v_1}$ be the velocity of the buggy after both man jump off simultaneously. For the closed system (two men + buggy), from the conservation of linear momentum,

$$M\vec{v_1} + 2m(\vec{u} + \vec{v_1}) = 0$$

or, $$\vec{v_1} = \frac{-2m\vec{u}}{M+2m} \qquad (1)$$

(ii) Let $\vec{v}'$ be the velocity of buggy with man, when one man jump off the buggy. For the closed system (buggy with one man + other man) from the conservation of linear momentum :

$$0 = (M+m)\vec{v}' + m(\vec{u} + \vec{v}') \qquad (2)$$

Let $\vec{v_2}$ be the sought velocity of the buggy when the second man jump off the buggy; then from conservation of linear momentum of the system (buggy + one man) :

$$(M+m)\vec{v}' = M\vec{v_2} + m(\vec{u} + \vec{v_2}) \qquad (3)$$

Solving equations (2) and (3) we get

$$\vec{v_2} = \frac{m(2M+3m)\vec{u}}{(M+m)(M+2m)} \qquad (4)$$

From (1) and (4)

$$\frac{v_2}{v_1} = 1 + \frac{m}{2(M+m)} > 1$$

Hence $v_2 > v_1$

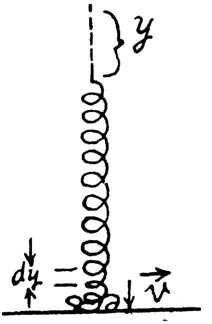
1.157 The descending part of the chain is in free fall, it has speed $v = \sqrt{2gh}$ at the instant, all its points have descended a distance y. The length of the chain which lands on the floor during the differential time interval dt following this instant is vdt.

For the incoming chain element on the floor :

From $dp_y = F_y\,dt$ (where y – axis is directed down)

$$0 - (\lambda\, v dt)\, v = F_y\, dt$$

or
$$F_y = -\lambda v^2 = -2\lambda g y$$



Hence, the force exerted on the falling chain equals λv^2 and is directed upward. Therefore from third law the force exerted by the falling chain on the table at the same instant of time becomes λv^2 and is directed downward.

Since a length of chain of weight (λyg) already lies on the table the total force on the floor is $(2\lambda yg) + (\lambda yg) = (3\lambda yg)$ or the weight of a length $3y$ of chain.

1.158 Velocity of the ball, with which it hits the slab, $v = \sqrt{2gh}$

After first impact, $v' = ev$ (upward) but according to the problem $v' = \frac{v}{\eta}$, so $e = \frac{1}{\eta}$ (1)

and momentum, imparted to the slab,

$$= mv - (-mv') = mv(1+e)$$

Similarly, velocity of the ball after second impact,

$$v'' = ev' = e^2 v$$

And momentum imparted $= m(v' + v'') = m(1+e)\,ev$

Again, momentum imparted during third impact,

$$= m(1+e)\,e^2 v, \text{ and so on,}$$

Hence, net momentum, imparted $= mv(1+e) + mve(1+e) + mve^2(1+e) + \ldots$

$$= mv(1+e)(1+e+e^2+\ldots)$$

$$= mv\frac{(1+e)}{(1-e)}, \text{ (from summation of G.P.)}$$

$$= \sqrt{2gh}\,\frac{\left(1+\frac{1}{\eta}\right)}{\left(1-\frac{1}{\eta}\right)} = m\sqrt{2gh}\,/\,(\eta+1)\,/\,(\eta-1) \text{ (Using Eq. 1)}$$

$$= 0{\cdot}2 \text{ kg m/s. (On substitution)}$$

1.159 (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

i.e. $\vec{r}_C =$ constant or, $\Delta\vec{r}_C = 0$ i.e. $\sum m_i\,\Delta\vec{r}_i = 0$

or,
$$m\left(\Delta\vec{r}_{mM} + \Delta\vec{r}_M\right) + M\,\Delta\vec{r}_M = 0$$

Thus,
$$m(\vec{l}' + \vec{l}) + M\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}'}{m+M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So,
$$0 = m[\vec{v}'(t) + \vec{v}_2(t)] + M\vec{v}_2(t)$$

1.159 (a) Since the resistance of water is negligibly small, the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

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Thus, $$m\,(\vec{l}\,' + \vec{l}) + M\,\vec{l} = 0, \text{ or, } \vec{l} = -\frac{m\vec{l}\,'}{m+M}$$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change,

So, $$0 = m\,[\,\vec{v}\,'(t) + \vec{v}_2(t)\,] + M\,\vec{v}_2(t)$$

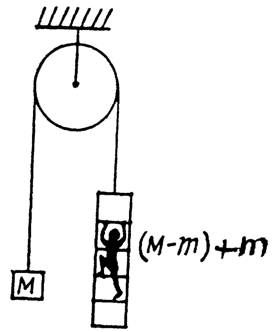
or, $$\vec{v}_2(t) = -\frac{m\,\vec{v}\,'(t)}{m+M} \qquad (1)$$

As $\vec{v}\,'(t)$ or $\vec{v}_2(t)$ is along horizontal direction, thus the sought force on the raft

$$= \frac{M\,d\vec{v}_2(t)}{dt} = -\frac{Mm}{m+M}\,\frac{d\vec{v}\,'(t)}{dt}$$

Note : *we may get the result of part* (a), *if we integrate Eq.* (1) *over the time of motion of man or raft.*

1.160 In the refrence frame fixed to the pulley axis the location of C.M. of the given system is described by the radius vector



$$\Delta\vec{r}_C = \frac{M\,\Delta\vec{r}_M + (M-m)\,\Delta\vec{r}_{(M-m)} + m\,\Delta\vec{r}_m}{2M}$$

But $\Delta\vec{r}_M = -\Delta\vec{r}_{(M-m)}$

and $\Delta\vec{r}_m = \Delta\vec{r}_{m(M-m)} + \Delta\vec{r}_{(M-m)}$

Thus $$\Delta\vec{r}_C = \frac{m\,\vec{l}\,'}{2M}$$

Note : *one may also solve this problem using momentum conservation.*

1.161 Velocity of cannon as well as that of shell equals $\sqrt{2\,gl\sin\alpha}$ down the inclined plane taken as the positive x – axis. From the linear impulse momentum theorem in projection form along x – axis for the system (connon + shell) i.e. $\Delta p_x = F_x\,\Delta t$:

$p\cos\alpha - M\sqrt{2\,gl\sin\alpha} = Mg\sin\alpha\,\Delta t$ (as mass of the shell is neligible)

or, $$\Delta t = \frac{p\cos\alpha - M\sqrt{2\,gl\sin\alpha}}{Mg\sin\alpha}$$

1.162 From conservation of momentum, for the system (bullet + body) along the initial direction of bullet

$$mv_0 = (m+M)\,v, \text{ or, } v = \frac{mv_0}{m+M}$$

1.163 When the disc breaks off the body M, its velocity towards right (along x-axis) equals the velocity of the body M, and let the disc's velocity in upward direction (along y-axis) at that moment be v'_y

From conservation of momentum, along x-axis for the system (disc + body)

$$mv = (m+M)\,v'_x \quad \text{or} \quad v'_x = \frac{mv}{m+M} \tag{1}$$

And from energy conservation, for the same system in the field of gravity :

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\,v'^2_x + \frac{1}{2}m\,v'^2_y + mgh',$$

where h' is the height of break off point from initial level. So,

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\frac{m^2v^2}{(M+m)}+\frac{1}{2}m\,v'^2_y + mgh', \qquad \text{using (1)}$$

or,
$$v'^2_y = v^2 - \frac{mv^2}{(m+M)} - 2\,gh'$$

Also, if h'' is the height of the disc, from the break-off point,

then,
$$v'^2_y = 2\,gh''$$

So,
$$2g\,(h''+h') = v^2 - \frac{m\,v^2}{(M+m)}$$

Hence, the total height, raised from the initial level

$$= h' + h'' = \frac{M\,v^2}{2g\,(M+m)}$$

1.164 (a) When the disc slides and comes to a plank, it has a velocity equal to $v = \sqrt{2\,gh}$. Due to friction between the disc and the plank the disc slows down and after some time the disc moves in one piece with the plank with velocity v' (say).

From the momentum conservation for the system (disc + plank) along horizontal towards right :

$$mv = (m+M)\,v' \quad \text{or} \quad v' = \frac{mv}{m+M}$$

Now from the equation of the increment of total mechanical energy of a system :

$$\frac{1}{2}(M+m)\,v'^2 - \frac{1}{2}mv^2 = A_{fr}$$

or,
$$\frac{1}{2}(M+m)\frac{m^2v^2}{(m+M)^2} - \frac{1}{2}mv^2 = A_{fr}$$

so,
$$\frac{1}{2}v^2\left[\frac{m^2}{M+m} - m\right] = A_{fr}$$

Hence,
$$A_{fr} = -\left(\frac{mM}{m+M}\right)gh = -\,\mu gh$$

$$\left(\text{where } \mu = \frac{mM}{m+M} = \text{reduced mass}\right)$$

(b) We look at the problem from a frame in which the hill is moving (together with the disc on it) to the right with speed u. Then in this frame the speed of the disc when it just gets onto the plank is, by the law of addition of velocities, $\bar{v} = u + \sqrt{2gh}$. Similarly the common speed of the plank and the disc when they move together is

$$\bar{v}' = u + \frac{m}{m+M}\sqrt{2gh}.$$

Then as above $\quad \bar{A}_{fr} = \frac{1}{2}(m+M)\bar{v}'^2 - \frac{1}{2}m\bar{v}^2 - \frac{1}{2}Mu^2$

$$= \frac{1}{2}(m+M)\left\{u^2 + \frac{2m}{m+M}u\sqrt{2gh} + \frac{m^2}{(m+M)^2}2gh\right\} - \frac{1}{2}(m+M)u^2 - \frac{1}{2}m\,2u\sqrt{2gh} - mgh$$

We see that $\bar{A}_{fr}$ is independent of u and is in fact just $-\mu g h$ as in (a). Thus the result obtained does not depend on the choice of reference frame.

Do note however that it will be in correct to apply "conservation of enegy" formula in the frame in which the hill is moving. The energy carried by the hill is not negligible in this frame. See alsó the next problem.

1.165 In a frame moving relative to the earth, one has to include the kinetic energy of the earth as well as earth's acceleration to be able to apply conservation of energy to the problem. In a reference frame falling to the earth with velocity v_0, the stone is initially going up with velocity v_0 and so is the earth. The final velocity of the stone is $0 = v_o - gt$ and that of the earth is $v_o + \frac{m}{M}gt$ (M is the mass of the earth), from Newton's third law, where $t =$ time of fall. From conservation of energy

$$\frac{1}{2}mv_0^2 + \frac{1}{2}Mv_0^2 + mgh = \frac{1}{2}M\left(v_0 + \frac{m}{M}v_0\right)^2$$

Hence $$\frac{1}{2}v_0^2\left(m + \frac{m^2}{M}\right) = mgh$$

Negecting $\frac{m}{M}$ in comparison with 1, we get

$$v_0^2 = 2gh \text{ or } v_o = \sqrt{2gh}$$

The point is this in earth's rest frame the effect of earth's accleration is of order $\frac{m}{M}$ and can be neglected but in a frame moving with respect to the earth the effect of earth's acceleration must be kept because it is of order one (i.e. large).

1.166 From conservation of momentum, for the closed system "both colliding particles"

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}$$

or, $$\vec{v} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{1(3\vec{i} - 2\vec{j}) + 2(4\vec{j} - 6\vec{k})}{3} = \vec{i} + 2\vec{j} - 4\vec{k}$$

Hence $$|\vec{v}| = \sqrt{1+4+16} \text{ m/s} = 4{\cdot}6 \text{ m/s}$$

1.167 For perfectly inelastic collision, in the C.M. frame, final kinetic energy of the colliding system (both spheres) becomes zero. Hence initial kinetic energy of the system in C.M. frame completely turns into the internal energy (Q) of the formed body. Hence

$$Q = \tilde{T}_i = \frac{1}{2}\mu\left|\vec{v}_1 - \vec{v}_2\right|^2$$

Now from energy conservation $\Delta T = -Q = -\frac{1}{2}\mu\left|\vec{v}_1 - \vec{v}_2\right|^2$,

In lab frame the same result is obtained as

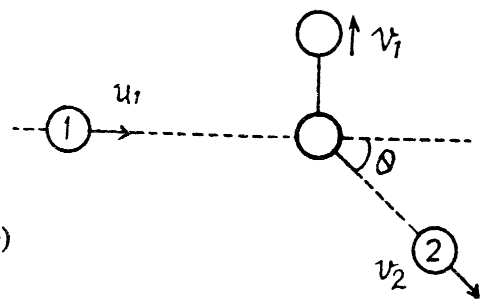
$$\Delta T = \frac{1}{2}\frac{(m_1\vec{v}_1 + m_2\vec{v}_2)^2}{m_1 + m_2} - \frac{1}{2}m_1\,|\vec{v}_1|^2 + m_2\,|\vec{v}_2|^2$$

$$= -\frac{1}{2}\mu\,|\vec{v}_1 - \vec{v}_2|^2$$

1.168 (a) Let the initial and final velocities of m_1 and m_2 are $\vec{u}_1$, $\vec{u}_2$ and $\vec{v}$, $\vec{v}_2$ respectively. Then from conservation of momentum along horizontal and vertical directions, we get :

$$m_1 u_1 = m_2 v_2 \cos\theta \qquad (1)$$

and $\quad m_1 v_1 = m_2 v_2 \sin\theta \qquad (2)$



Squaring (1) and (2) and then adding them,

$$m_2^2 v_2^2 = m_1^2(u_1^2 + v_1^2)$$

Now, from kinetic energy conservation,

$$\frac{1}{2}m_1 u_1^2 = \frac{1}{2}m_2 v_2^2 + \frac{1}{2}m_1 v_1^2 \qquad (3)$$

or, $\quad m(u_1^2 - v_1^2) = m_2 v_2^2 = m_2\frac{m_1^2}{m_2^2}(u_1^2 + v_1^2)$ [Using (3)]

or, $$u_1^2\left(1 - \frac{m_1}{m_2}\right) = v_1^2\left(1 + \frac{m_1}{m_2}\right)$$

or, $$\left(\frac{v_1}{u_1}\right)^2 = \frac{m_2 - m_1}{m_1 + m_2} \qquad (4)$$

So, fraction of kinetic energy lost by the particle 1,

$$= \frac{\frac{1}{2}m_1 u_1^2 - \frac{1}{2}m_1 v_1^2}{\frac{1}{2}m_1 u_1^2} = 1 - \frac{v_1^2}{u_1^2}$$

$$= 1 - \frac{m_2 - m_1}{m_1 + m_2} = \frac{2\,m_1}{m_1 + m_2} \quad \text{[Using (4)]} \qquad (5)$$

(b) When the collision occurs head on,

$$m_1 u_1 = m_1 v_1 + m_2 v_2 \qquad (1)$$

and from conservation of kinetic energy,

$$\frac{1}{2}m_1 u_1^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2\left[\frac{m_1(u_1 - v_1)^2}{m_2}\right]^2 \text{ [Using (5)]}$$

or, $$v_1\left(1 + \frac{m_1}{m_2}\right) = u_1\left(\frac{m_1}{m_2} - 1\right)$$

or, $$\frac{v_1}{u_1} = \frac{(m_1/m_2 - 1)}{(1 + m_1/m_2)} \quad (6)$$

Fraction of kinetic energy, lost

$$= 1 - \frac{v_1^2}{u_1^2} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2 = \frac{4\,m_1 m_2}{(m_1 + m_2)^2} \text{ [Using (6)]}$$

1.169 (a) When the particles fly apart in opposite direction with equal velocities (say v), then from conservatin of momentum,

$$m_1 u + 0 = (m_2 - m_1)\,v \quad (1)$$

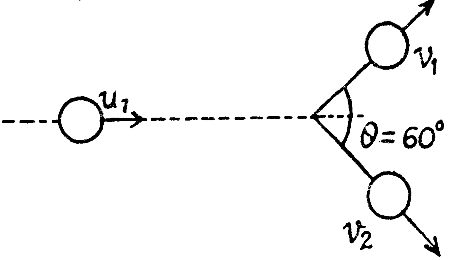
and from conservation of kinetic energy,

$$\frac{1}{2}m_1 u^2 = \frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2$$

or, $$m_1 u^2 = (m_1 + m_2)\,v^2 \quad (2)$$

From Eq. (1) and (2),

$$m_1 u^2 = (m_1 + m_2)\frac{m_1^2 u^2}{(m_2 - m_1)^2}$$



or, $$m_2^2 - 3\,m_1 m_2 = 0$$

Hence $$\frac{m_1}{m_2} = \frac{1}{3} \text{ as } m_2 \neq 0$$

(b) When they fly apart symmetrically relative to the initial motion direction with the angle of divergence $\theta = 60°$,

From conservation of momentum, along horizontal and vertical direction,

$$m_1 u_1 = m_1 v_1 \cos(\theta/2) + m_2 v_2 \cos(\theta/2) \quad (1)$$

and $$m_1 v_1 \sin(\theta/2) = m_2 v_2 \sin(\theta/2)$$

or, $$m_1 v_1 = m_2 v_2 \quad (2)$$

Now, from conservation of kinetic energy,

$$\frac{1}{2}m_1 u_1^2 + 0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad (3)$$

From (1) and (2),

$$m_1 u_1 = \cos(\theta/2)\left(m_1 v_1 + \frac{m_1 v_1}{m_2}m_2\right) = 2\,m_1 v_1 \cos(\theta/2)$$

So, $$u_1 = 2 v_1 \cos(\theta/2) \quad (4)$$
From (2), (3), and (4)

$$4 m_1 \cos^2(\theta/2) v_1^2 = m_1 v_1^2 + \frac{m_2 m_1^2 v_1^2}{m_2^2}$$

$$\text{or,} \quad 4 \cos^2(\theta/2) = 1 + \frac{m_1}{m_2}$$

or, $$\frac{m_1}{m_2} = 4 \cos^2 \frac{\theta}{2} - 1$$

and putting the value of θ, we get, $\frac{m_1}{m_2} = 2$

1.170 If (v_{1x}, v_{1y}) are the instantaneous velocity components of the incident ball and (v_{2x}, v_{2y}) are the velocity components of the struck ball at the same moment, then since there are no external impulsive forces (i.e. other than the mutual interaction of the balls)
We have $$u \sin\alpha = v_{1y} \;, \quad v_{2y} = 0$$

$$m u \cos\alpha = m v_{1x} + m v_{2x}$$

The impulsive force of mutual interaction satisfies

$$\frac{d}{dt}(v_{1x}) = \frac{F}{m} = -\frac{d}{dt}(v_{2x})$$

(F is along the x axis as the balls are smooth. Thus Y component of momentum is not transferred.) Since loss of K.E. is stored as deformation energy D, we have

$$D = \frac{1}{2} m u^2 - \frac{1}{2} m v_1^2 - \frac{1}{2} m v_2^2$$

$$= \frac{1}{2} m u^2 \cos^2\alpha - \frac{1}{2} m v_{1x}^2 - \frac{1}{2} m v_{2x}^2$$

$$= \frac{1}{2m}\left[m^2 u^2 \cos^2\alpha - m^2 v_{1x}^2 - (m u \cos\alpha - m v_{1x})^2 \right]$$

$$= \frac{1}{2m}\left[2m^2 u \cos\alpha\, v_{1x} - 2m^2 v_{1x}^2 \right] = m(v_{1x} u \cos\alpha - v_{1x}^2)$$

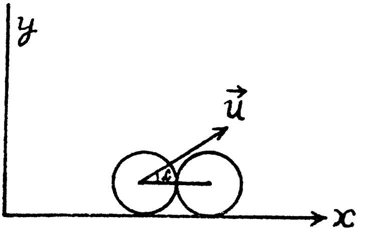
$$= m\left[\frac{u^2\cos^2\alpha}{4} - \left(\frac{u\cos\alpha}{2} - v_{1x} \right)^2 \right]$$

We see that D is maximum when

$$\frac{u\cos\alpha}{2} = v_{1x}$$

and $$D_{max} = \frac{m u^2 \cos^2\alpha}{4}$$

Then $$\eta = \frac{D_{max}}{\frac{1}{2} m u^2} = \frac{1}{2}\cos^2\alpha = \frac{1}{4}$$



On substiuting $\alpha = 45°$

1.171 From the conservation of linear momentum of the shell just before and after its fragmentation

$$3\vec{v} = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \quad (1)$$

where $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the velocities of its fragments.

From the energy conservation $\quad 3\eta v^2 = v_1^2 + v_2^2 + v_3^2 \quad (2)$

Now $$\vec{\tilde{v}}_i \text{ or } \vec{v}_{iC} = \vec{v}_i - \vec{v}_C = \vec{v}_i - \vec{v} \quad (3)$$

where $\vec{v}_C = \vec{v} =$ velocity of the C.M. of the fragments the velocity of the shell. Obviously in the C.M. frame the linear momentum of a system is equal to zero, so

$$\vec{\tilde{v}}_1 + \vec{\tilde{v}}_2 + \vec{\tilde{v}}_3 = 0 \quad (4)$$

Using (3) and (4) in (2), we get

$$3\eta v^2 = (\vec{v} + \vec{\tilde{v}}_1)^2 + (\vec{v} + \vec{\tilde{v}}_2)^2 + (\vec{v} - \vec{\tilde{v}}_1 - \vec{\tilde{v}}_2)^2 \quad = 3v^2 + 2\tilde{v}_1^2 + 2\tilde{v}_2^2 + 2\,\vec{\tilde{v}}_1 \cdot \vec{\tilde{v}}_2$$

or, $$2\tilde{v}_1^2 + 2\tilde{v}_1\tilde{v}_2\cos\theta + 2\tilde{v}_2^2 + 3(1-\eta)v^2 = 0 \quad (5)$$

If we have had used $\vec{\tilde{v}}_2 = -\vec{\tilde{v}}_1 - \vec{\tilde{v}}_3$, then Eq. 5 were contain $\tilde{v}_3$ instead of $\tilde{v}_2$ and so on.

The problem being symmetrical we can look for the maximum of any one. Obviously it will be the same for each.

For $\tilde{v}_1$ to be real in Eq. (5)

$$4\tilde{v}_2^2\cos^2\theta \geq 8(2\tilde{v}_2^2 + 3(1-\eta)v^2) \text{ or } 6(\eta - 1)v^2 \geq (4 - \cos^2\theta)\tilde{v}_2^2$$

So, $$\tilde{v}_2 \leq v\sqrt{\frac{6(\eta-1)}{4-\cos^2\theta}} \quad \text{or} \quad \tilde{v}_{2(\max)} = \sqrt{2(\eta-1)}\ v$$

Hence $$v_{2(\max)} = |\vec{v} + \vec{\tilde{v}}_2|_{\max} = v + \sqrt{2(\eta-1)}\ v = v\left(1 + \sqrt{2(\eta-1)}\right) = 1 \text{ km/s}$$

Thus owing to the symmetry

$$v_{1(\max)} = v_{2(\max)} = v_{3(\max)} = v\left(1 + \sqrt{2(\eta-1)}\right) = 1 \text{ km/s}$$

1.172 Since, the collision is head on, the particle 1 will continue moving along the same line as before the collision, but there will be a change in the magnitude of it's velocity vector. Let it starts moving with velocity v_1 and particle 2 with v_2 after collision, then from the conservation of momentum

$$mu = mv_1 + mv_2 \quad \text{or,} \quad u = v_1 + v_2 \quad (1)$$

And from the condition, given,

$$\eta = \frac{\frac{1}{2}mu^2 - \left(\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2\right)}{\frac{1}{2}mu^2} = 1 - \frac{v_1^2 + v_2^2}{u^2}$$

or, $$v_1^2 + v_2^2 = (1-\eta)u^2 \quad (2)$$

From (1) and (2),

$$v_1^2 + (u - v_1)^2 = (1-\eta)u^2$$

or, $$v_1^2 + u^2 - 2uv_1 + v_1^2 = (1-\eta)u^2$$

or, $$2v_1^2 - 2v_1 u + \eta u^2 = 0$$

So, $$v_1 = 2u \pm \frac{\sqrt{4u^2 - 8\eta u^2}}{4}$$

$$= \frac{1}{2}\left[u \pm \sqrt{u^2 - 2\eta u^2}\right] = \frac{1}{2}u\,(1 \pm \sqrt{1 - 2\eta}\,)$$

Positive sign gives the velocity of the 2nd particle which lies ahead. The negative sign is correct for v_1.

So, $v_1 = \frac{1}{2}u\,(1 - \sqrt{1 - 2\eta}\,) = 5$ m/s will continue moving in the same direction.

Note that $v_1 = 0$ if $\eta = 0$ as it must.

1.173 Since, no external impulsive force is effective on the system "$M + m$", its total momentum along any direction will remain conserved.

So from $p_x =$ const.

$$mu = Mv_1 \cos\theta \quad \text{or,} \quad v_1 = \frac{m}{M}\frac{u}{\cos\theta} \tag{1}$$

and from $p_y =$ const

$$mv_2 = Mv_1 \sin\theta \quad \text{or,} \quad v_2 = \frac{M}{m}v_1 \sin\theta = u\tan\theta, \quad \text{[using (1)]}$$

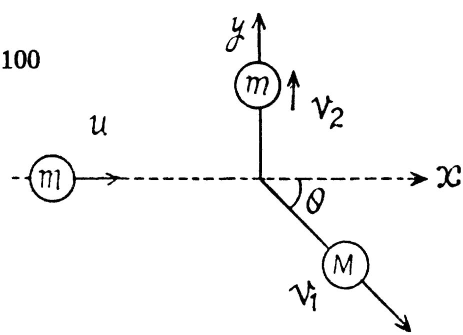
Final kinetic energy of the system

$$T_f = \frac{1}{2}mv_2^2 + \frac{1}{2}Mv_1^2$$

And initial kinetic energy of the system $= \frac{1}{2}mu^2$

So, $$\%\ \text{change} = \frac{T_f - T_i}{T_i} \times 100$$

$$= \frac{\frac{1}{2}mu^2\tan^2\theta + \frac{1}{2}M\frac{m^2}{M^2}\frac{u^2}{\cos^2\theta} - \frac{1}{2}mu^2}{\frac{1}{12}mu^2} \times 100$$



$$= \frac{\frac{1}{2}u^2\tan^2\theta + \frac{1}{2}\frac{m}{M}u^2\sec^2\theta - \frac{1}{2}u^2}{\frac{1}{2}u^2} \times 100$$

$$= \left(\tan^2\theta + \frac{m}{M}\sec^2\theta - 1\right) \times 100$$

and putting the values of θ and $\frac{m}{M}$, we get % of change in kinetic energy $= -40\ \%$

1.174 (a) Let the particles m_1 and m_2 move with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively. On the basis of solution of problem 1.147 (b)

$$\tilde{p} = \mu\, v_{rel} = \mu\, \left|\vec{v_1} - \vec{v_2}\right|$$

As $\qquad \vec{v_1} \perp \vec{v_2}$

So, $\qquad \tilde{p} = \mu\sqrt{v_1^2 + v_2^2}$ where $\mu = \frac{m_1 m_2}{m_1 + m_2}$

(b) Again from 1.147 (b)

$$\tilde{T} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\mu \left|\vec{v_1} - \vec{v_2}\right|^2$$

So, $\qquad \tilde{T} = \frac{1}{2}\mu\,(v_1^2 + v_2^2)$

1.175 From conservation of momentum

$$\vec{p_1} = \vec{p_1}' + \vec{p_2}'$$

so $\qquad \left(\vec{p_1} - \vec{p_1}'\right)^2 = p_1^2 - 2p_1\, p_1' \cos\theta_1 + p_1'^2 = p_2'^2$

From conservation of energy

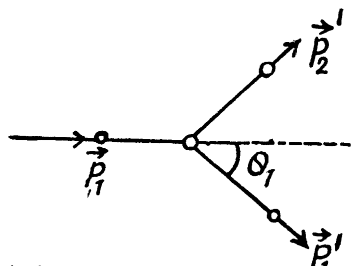
$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

Eliminating p_2' we get

$$0 = p_1'^2\left(1 + \frac{m_2}{m_1}\right) - 2p_1' p_1 \cos\theta_1 + p_1^2\left(1 - \frac{m_2}{m_1}\right)$$

This quadratic equation for p_1' has a real solution in terms of p_1 and $\cos\theta_1$ only if

$$4\cos^2\theta_1 \geq 4\left(1 - \frac{m_2^2}{m_1^2}\right)$$



or $\qquad \sin^2\theta_1 \leq \frac{m_2^2}{m_1^2}$

or $\quad \sin\theta_1 \leq +\frac{m_2}{m_1} \quad$ or $\quad \sin\theta_1 \geq -\frac{m_2}{m_1}$

This clearly implies (since only + sign makes sense) that

$$\sin\theta_{1\,max} = \frac{m_2}{m_1}.$$

1.176 From the symmetry of the problem, the velocity of the disc A will be directed either in the initial direction or opposite to it just after the impact. Let the velocity of the disc A after the collision be v' and be directed towards right after the collision. It is also clear from the symmetry of problem that the discs B and C have equal speed (say v'') in the directions, shown. From the condition of the problem,

$$\cos\theta = \frac{\eta\frac{d}{2}}{d} = \frac{\eta}{2} \text{ so, } \sin\theta = \sqrt{4 - \eta^2}\,/\,2 \qquad (1)$$

For the three discs, system, from the conservation of linear momentum in the symmetry direction (towards right)

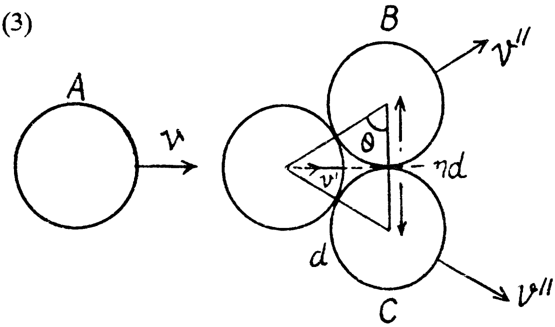
$$mv = 2m\,v''\sin\theta + m\,v' \text{ or, } v = 2\,v''\sin\theta + v' \qquad (2)$$

From the definition of the coefficeint of restitution, we have for the discs A and B (or C)

$$e = \frac{v'' - v' \sin \theta}{v \sin \theta - 0}$$

But $e = 1$, for perfectly elastic collision,

So, $v \sin \theta = v'' - v' \sin \theta$ (3)



From (2) and (3),

$$v' = \frac{v(1 - 2\sin^2 \theta)}{(1 + 2\sin^2 \theta)}$$

$$= \frac{v(\eta^2 - 2)}{6 - \eta^2} \quad \{\text{using (1)}\}$$

Hence we have,

$$v' = \frac{v(\eta^2 - 2)}{6 - \eta^2}$$

Therefore, the disc A will recoil if $\eta < \sqrt{2}$ and stop if $\eta = \sqrt{2}$.

Note : *One can write the equations of momentum conservation along the direction perpendicular to the initial direction of disc A and the consevation of kinetic energy instead of the equation of restitution.*

1.177 (a) Let a molecule comes with velocity $\vec{v}_1$ to strike another stationary molecule and just after collision their velocities become $\vec{v}'_1$ and $\vec{v}'_2$ respectively. As the mass of the each molecule is same, conservation of linear momentum and conservation of kinetic energy for the system (both molecules) respectively gives :

$$\vec{v}_1 = \vec{v}'_1 + \vec{v}'_2$$

and

$$v_1^2 = v_1'^2 + v_2'^2$$

From the property of vector addition it is obvious from the obtained Eqs. that

$$\vec{v}'_1 \perp \vec{v}'_2 \quad \text{or} \quad \vec{v}'_1 \cdot \vec{v}'_2 = 0$$

(b) Due to the loss of kinetic energy in inelastic collision $v_1^2 > v_1'^2 + v_2'^2$

so, $\vec{v}'_1 \cdot \vec{v}'_2 > 0$ and therefore angle of divergence $< 90°$.

1.178 Suppose that at time t, the rocket has the mass m and the velocity $\vec{v}$, relative to the reference frame, employed. Now consider the inertial frame moving with the velocity that the rocket has at the given moment. In this reference frame, the momentum increament that the rocket & ejected gas system acquires during time dt is,

$$d\vec{p} = m\,d\vec{v} + \mu\,dt\,\vec{u} = \vec{F}\,dt$$

or,

$$m\frac{d\vec{v}}{dt} = \vec{F} - \mu\vec{u}$$

or,

$$m\vec{w} = \vec{F} - \mu\vec{u}$$

1.179 According to the question, $\vec{F} = 0$ and $\mu = -dm/dt$ so the equation for this system becomes,

$$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $d\vec{v} \uparrow\downarrow \vec{u}$ so, $m\,dv = -u\,dm$.

Integrating within the limits :

$$\frac{1}{u}\int_0^v dv = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or} \quad \frac{v}{u} = \ln\frac{m_0}{m}$$

Thus, $v = u\ln\frac{m_0}{m}$

As $d\vec{v} \uparrow\downarrow \vec{u}$, so in vector form $\vec{v} = -\vec{u}\ln\frac{m_0}{m}$

1.180 According to the question, $\vec{F}$ (external force) = 0

So, $$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u}$$

As $$d\vec{v} \uparrow\downarrow \vec{u},$$

so, in scalar form, $$m\,dv = -u\,dm$$

or, $$\frac{wdt}{u} = -\frac{dm}{m}$$

Integrating within the limits for $m(t)$

$$\frac{wt}{u} = -\int_{m_0}^{m}\frac{dm}{m} \quad \text{or,} \quad \frac{v}{u} = -\ln\frac{m}{m_0}$$

Hence, $$m = m_0\,e^{-(wt/u)}$$

1.181 As $\vec{F} = 0$, from the equation of dynamics of a body with variable mass;

$$m\frac{d\vec{v}}{dt} = \vec{u}\frac{dm}{dt} \quad \text{or,} \quad d\vec{v} = \vec{u}\frac{dm}{m} \tag{1}$$

Now $d\vec{v} \uparrow\downarrow \vec{u}$ and since $\vec{u} \perp \vec{v}$, we must have $|d\vec{v}| = v_0\,d\alpha$ (because v_0 is constant) where $d\alpha$ is the angle by which the spaceship turns in time dt.

So, $$-u\frac{dm}{m} = v_0\,d\alpha \quad \text{or,} \quad d\alpha = -\frac{u}{v_0}\frac{dm}{m}$$

or, $$\alpha = -\frac{u}{v_0}\int_{m_0}^{m}\frac{dm}{m} = \frac{u}{v_0}\ln\left(\frac{m_0}{m}\right)$$

1.182 We have $\frac{dm}{dt} = -\mu$ or, $dm = -\mu\, dt$

Integrating $$\int_{m_0}^{m} dm = -\mu \int_0^t dt \text{ or, } m = m_0 - \mu t$$

As $\vec{u} = 0$ so, from the equation of variable mass system :

$$(m_0 - \mu t)\frac{d\vec{v}}{dt} = \vec{F} \text{ or, } \frac{d\vec{v}}{dt} = \vec{w} = \vec{F}/(m_0 - \mu t)$$

or, $$\int_0^{\vec{v}} d\vec{v} = \vec{F}\int_0^t \frac{dt}{(m_0 - \mu t)}$$

Hence $$\vec{v} = \frac{\vec{F}}{\mu} \ln\left(\frac{m_0}{m_0 - \mu t}\right)$$

1.183 Let the car be moving in a reference frame to which the hopper is fixed and at any instant of time, let its mass be m and velocity $\vec{v}$.

Then from the general equation, for variable mass system.

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{u}\frac{dm}{dt}$$

We write the equation, for our system as,

$$m\frac{d\vec{v}}{dt} = \vec{F} - \vec{v}\frac{dm}{dt} \text{ as, } \vec{u} = -\vec{v} \qquad (1)$$

So $$\frac{d}{dt}(m\vec{v}) = \vec{F}$$

and $$\vec{v} = \frac{\vec{F}t}{m} \text{ on integration.}$$

But $$m = m_0 + \mu t$$

so, $$\vec{v} = \frac{\vec{F}t}{m_0\left(1 + \frac{\mu t}{m_0}\right)}$$

Thus the sought acceleration, $$\vec{w} = \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m_0\left(1 + \frac{\mu t}{m_0}\right)^2}$$

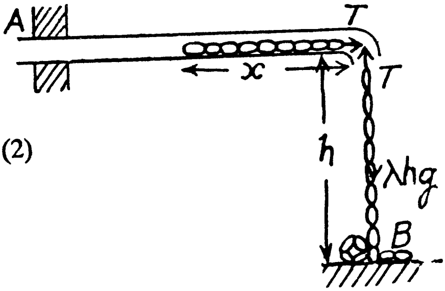
1.184 Let the length of the chain inside the smooth horizontal tube at an arbitrary instant is x.

From the equation,

$$m\vec{w} = \vec{F} + \vec{u}\frac{dm}{dt}$$

as $\vec{u} = 0$, $\vec{F} \uparrow\uparrow \vec{w}$, for the chain inside the tube

$$\lambda x w = T \text{ where } \lambda = \frac{m}{l} \qquad (1)$$



Similarly for the overhanging part,

$$\vec{u} = 0$$

Thus $\qquad mw = F$

or $\qquad \lambda h w = \lambda h g - T \qquad (2)$

From (1) and (2),

$$\lambda (x + h) w = \lambda h g \quad \text{or,} \quad (x + h) v \frac{dv}{ds} = hg$$

or,
$$(x + h) v \frac{dv}{(-dx)} = gh,$$

[As the length of the chain inside the tube decreases with time, $ds = -dx$.]

or,
$$v\,dv = -gh \frac{dx}{x + h}$$

Integrating,
$$\int_0^v v\,dv = -gh \int_{(l-h)}^0 \frac{dx}{x + h}$$

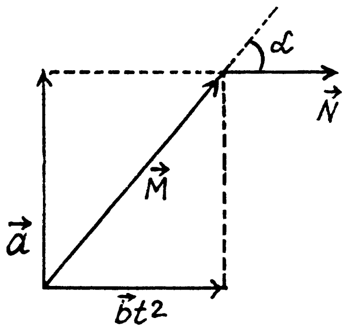
or,
$$\frac{v^2}{2} = gh \ln\left(\frac{l}{h}\right) \quad \text{or} \quad v = \sqrt{2gh \ln\left(\frac{l}{h}\right)}$$

1.185 Force moment relative to point O ;

$$\vec{N} = \frac{d\vec{M}}{dt} = 2\vec{bt}$$

Let the angle between $\vec{M}$ and $\vec{N}$,

$\alpha = 45°$ at $t = t_0$,



$$\text{Then } \frac{1}{\sqrt{2}} = \frac{\vec{M} \cdot \vec{N}}{|\vec{M}||\vec{N}|} = \frac{(\vec{a} + \vec{b} t_0^2) \cdot (2\vec{b} t_0)}{\sqrt{a^2 + b^2 t_0^4}\; 2bt_0}$$

$$= \frac{2b^2 t_0^3}{\sqrt{a^2 + b^2 t_0^4}\; 2bt_0} = \frac{b t_0^2}{\sqrt{a^2 + b^2 t_0^4}}$$

So, $\quad 2 b^2 t_0^4 = a^2 + b^2 t_0^4$ or, $t_0 = \sqrt{\frac{a}{b}}$ (as t_0 cannot be negative)

It is also obvious from the figure that the angle α is equal to 45° at the moment t_0, when $a = b t_0^2$, i.e. $t_0 = \sqrt{a/b}$ and $\vec{N} = 2\sqrt{\frac{a}{b}}\,\vec{b}$.

1.186 $\vec{M}(t) = \vec{r} \times \vec{p} = \left(\vec{v_0}t + \frac{1}{2}\vec{g}t^2\right) \times m\left(\vec{v_0} + \vec{g}t\right)$

$$= mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(-\vec{k}) + \frac{1}{2}mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(\vec{k})$$

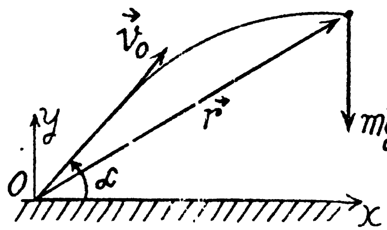
$$= \frac{1}{2}mv_0 gt^2 \cos\alpha\,(-\vec{k}):$$

Thus $M(t) = \dfrac{mv_0 g t^2 \cos\alpha}{2}$

Thus angular momentum at maximum height

i.e. at $t = \dfrac{\tau}{2} = \dfrac{v_0 \sin\alpha}{g}$,

$M\left(\dfrac{\tau}{2}\right) = \left(\dfrac{m v_0^3}{2g}\right)\sin^2\alpha\cos\alpha = 37\text{ kg} - \text{m}^2/\text{s}$

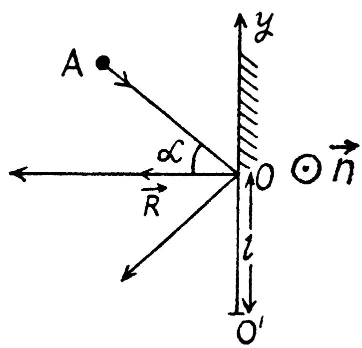


Alternate :

$$\vec{M}(0) = 0 \text{ so, } \vec{M}(t) = \int_0^t \vec{N}\,dt = \int_0^t (\vec{r} \times m\vec{g})$$

$$= \int_0^t \left[\left(\vec{v_0}t + \frac{1}{2}\vec{g}t^2\right) \times m\vec{g}\right] dt = \left(\vec{v_0} \times m\vec{g}\right)\frac{t^2}{2}$$

1.187 (a) The disc experiences gravity, the force of reaction of the horizontal surface, and the force $\vec{R}$ of reaction of the wall at the moment of the impact against it. The first two forces counter-balance each other, leaving only the force $\vec{R}$. It's moment relative to any point of the line along which the vector $\vec{R}$ acts or along normal to the wall is equal to zero and therefore the angular momentum of the disc relative to any of these points does not change in the given process.



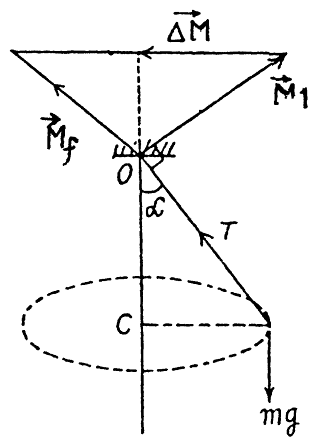
(b) During the course of collision with wall the position of disc is same and is equal to $\vec{r}_{oo'}$. Obviously the increment in linear momentum of the ball $\Delta\vec{p} = 2mv\cos\alpha\,\hat{n}$

Here, $\Delta\vec{M} = \vec{r}_{oo'} \times \Delta\vec{p} = 2\,mv\,l\cos\alpha\,\hat{n}$ and directed normally emerging from the plane of figure

Thus $|\Delta\vec{M}| = 2\,mv\,l\cos\alpha$

1.188 (a) The ball is under the influence of forces $\vec{T}$ and $m\vec{g}$ at all the moments of time, while moving along a horizontal circle. Obviously the vertical component of $\vec{T}$ balance $m\vec{g}$ and so the net moment of these two about any point becoems zero. The horizontal component of $\vec{T}$, which provides the centripetal acceleration to ball is already directed toward the centre (C) of the horizontal circle, thus its moment about the point C equals zero at all the moments of time. Hence the net moment of the force acting on the ball about point C equals zero and that's why the angular mommetum of the ball is conserved about the horizontal circle.

(b) Let α be the angle which the thread forms with the vertical.



Now from equation of particle dynamics :

$T\cos\alpha = mg$ and $T\sin\alpha = m\omega^2 l\sin\alpha$

Hence on solving $\cos\alpha = \dfrac{g}{\omega^2 l}$ (1)

As $|\vec{M}|$ is constant in magnitude so from figure.

$|\Delta\vec{M}| = 2M\cos\alpha$ where

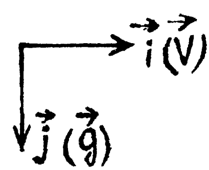
$M = |\vec{M}_i| = |\vec{M}_f|$

$= |\vec{r}_{bo}\times m\vec{v}| = mv\,l\,(\text{as } \vec{r}_{bo}\perp\vec{v})$

Thus $|\Delta\vec{M}| = 2mv\,l\cos\alpha = 2m\omega\,l^2\sin\alpha\cos\alpha$

$$= \frac{2mgl}{\omega}\sqrt{1-\left(\frac{g}{\omega^2 t}\right)^2} \text{ (using 1).}$$

1.189 During the free fall time $t = \tau = \sqrt{\dfrac{2h}{g}}$, the reference point O moves in hoizontal direction (say towards right) by the distance $V\tau$. In the translating frame as $\vec{M}(O) = 0$, so



$$\Delta\vec{M} = \vec{M}_f = \vec{r}$$

$$= (-V\tau\vec{i}+h\vec{j})\times m[g\tau\vec{j}-V\vec{i}]$$

$$= -mVg\tau^2\vec{h}+mVh(+\vec{k})$$

$$= -mVg\left(\frac{2h}{g}\right)\vec{k}+mVh(+\vec{k}) = -mVh\vec{k}$$

Hence $|\Delta\vec{M}| = mVh$

1.190 The Coriolis force is $(2m\vec{v}'\times\vec{\omega})$.

Here $\vec{\omega}$ is along the z-axis (vertical). The moving disc is moving with velocity v_0 which is constant. The motion is along the x-axis say. Then the Coriolis force is along y-axis and has the magnitude $2m\,v_0\,\omega$. At time t, the distance of the centre of moving disc from O is $v_0 t$ (along x-axis). Thus the torque N due to the coriolis force is

$N = 2m\,v_0\,\omega\cdot v_0 t$ along the z-axis.

Hence equating this to $\frac{dM}{dt}$

$$\frac{dM}{dt} = 2m\,v_0^2\,\omega t \quad \text{or} \quad M = mv_0^2\,\omega t^2 + \text{constant}.$$

The constant is irrelevant and may be put equal to zero if the disc is originally set in motion from the point O.

This discussion is approximate. The Coriolis force will cause the disc to swerve from straight line motion and thus cause deviation from the above formula which will be substantial for large t.

1.191 If $\dot{r}$ = radial velocity of the particle then the total energy of the particle at any instant is

$$\frac{1}{2}m\,\dot{r}^2 + \frac{M^2}{2mr^2} + kr^2 = E \tag{1}$$

where the second term is the kinetic energy of angular motion about the centre O. Then the extreme values of r are determined by $\dot{r} = 0$ and solving the resulting quadratic equation

$$k(r^2)^2 - Er^2 + \frac{M^2}{2m} = 0$$

we get

$$r^2 = \frac{E \pm \sqrt{E^2 - \frac{2M^2k}{m}}}{2k}$$

From this we see that

$$E = k(r_1^2 + r_2^2) \tag{2}$$

where r_1 is the minimum distance from O and r_2 is the maximum distance. Then

$$\frac{1}{2}mv_2^2 + 2kr_2^2 = k\,(\,r_1^2 + r_2^2\,)$$

Hence, $$m = \frac{2kr^2}{v_2^2}$$

·**Note** : Eq. (1) can be derived from the standard expression for kinetic energy and angular momentum in plane poler coordinates :

$$T = \frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}m\,r^2\,\dot{\theta}^2$$

M = angular momentum = $mr^2\,\dot{\theta}$

1.192 The swinging sphere experiences two forces : The gravitational force and the tension of the thread. Now, it is clear from the condition, given in the problem, that the moment of these forces about the vertical axis, passing through the point of suspension $N_z = 0$. Consequently, the angular momentum M_z of the sphere relative to the given axis (z) is constant.

Thus $$mv_0\,(l\sin\theta) = mv\,l \tag{1}$$

where m is the mass of the sphere and v is it s velocity in the position, when the thread forms an angle $\frac{\pi}{2}$ with the vertical. Mechanical energy is also conserved, as the sphere is under the influence if only one other force, i.e. tension, which does not perform any work, as it is always perpendicular to the velocity.

So, $$\frac{1}{2}mv_0^2 + mg\,l\cos\theta = \frac{1}{2}mv^2 \qquad (2)$$

From (1) and (2), we get,

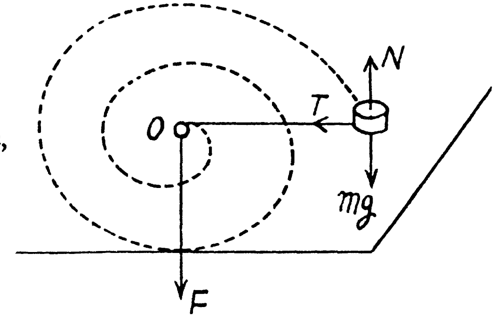
$$v_0 = \sqrt{2gl/\cos\theta}$$

1.193 Forces, acting on the mass m are shown in the figure. As $\vec{N} = m\vec{g}$, the net torque of these two forces about any fixed point must be equal to zero. Tension T, acting on the mass m is a central force, which is always directed towards the centre O. Hence the moment of force T is also zero about the point O and therefore the angular momentum of the particle m is conserved about O.

Let, the angular velocity of the particle be ω, when the separation between hole and particle m is r, then from the conservation of momentum about the point O, :

$$m(\omega_0 r_0)\,r_0 = m(\omega r)\,r,$$

or $$\omega = \frac{\omega_0 r_0^2}{r^2}$$



Now, from the second law of motion for m,

$$T = F = m\,\omega^2 r$$

Hence the sought tension;

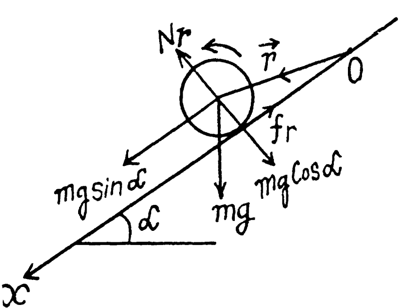
$$F = \frac{m\,\omega_0^2 r_0^4 r}{r^4} = \frac{m\,\omega_0^2 r_0^4}{r^3}$$

1.194 On the given system the weight of the body m is the only force whose moment is effective about the axis of pulley. Let us take the sense of $\vec{\omega}$ of the pulley at an arbitrary instant as the positive sense of axis of rotation (z-axis)

As $$M_z(0) = 0, \text{ so, } \Delta M_z = M_z(t) = \int N_z\,dt$$

So, $$M_z(t) = \int_0^t mgR\,dt = mgRt$$

1.195 Let the point of contact of sphere at initial moment ($t = 0$) be at O. At an arbitrary moment, the forces acting on the sphere are shown in the figure. We have normal reaction $N_r = mg\sin\alpha$ and both pass through same line and the force of static friction passes through the point O, thus the moment about point O becomes zero. Hence $mg\sin\alpha$ is the only force which has effective torque about point O, and is given by $|\vec{N}| = mgR\sin\alpha$ normally emerging from the plane of figure.



As $\vec{M}(t=0) = 0$, so, $\Delta\vec{M} = \vec{M}(t) = \int \vec{N}\,dt$

Hence, $M(t) = Nt = mgR\sin\alpha t$

1.196 Let position vectors of the particles of the system be $\vec{r}_i$ and $\vec{r}_i\,'$ with respect to the points O and O' respectively. Then we have,

$$\vec{r}_i = \vec{r}_i\,' + \vec{r}_0 \qquad (1)$$

where $\vec{r}_0$ is the radius vector of O' with respect to O.

Now, the angular momentum of the system relative to the point O can be written as follows;

$$\vec{M} = \sum \left(\vec{r}_i \times \vec{p}_i\right) = \sum \left(\vec{r}_i\,' \times \vec{p}_i\right) + \sum \left(\vec{r}_0 \times \vec{p}_i\right) \qquad \text{[using (1)]}$$

or, $$\vec{M} = \vec{M}\,' + \left(\vec{r}_0 \times \vec{p}\right), \text{ where, } \vec{p} = \sum \vec{p}_i \qquad (2)$$

From (2), if the total linear momentum of the system, $\vec{p} = 0$, then its angular momentum does not depend on the choice of the point O.

Note that in the C.M. frame, the system of particles, as a whole is at rest.

1.197 On the basis of solution of problem 1.196, we have concluded that; "in the C.M. frame, the angular momentum of system of particles is independent of the choice of the point, relative to which it is determined" and in accordance with the problem, this is denoted by $\vec{M}$.

We denote the angular momentum of the system of particles, relative to the point O, by $\vec{M}$. Since the internal and proper angular momentum $\tilde{M}$, in the C.M. frame, does not depend on the choice of the point O', this point may be taken coincident with the point O of the K-frame, at a given moment of time. Then at that moment, the radius vectors of all the particles, in both reference frames, are equal $(\vec{r}_i\,' = \vec{r}_i)$ and the velocities are related by the equation,

$$\vec{v}_i = \vec{\tilde{v}}_i + \vec{v}_c\,, \qquad (1)$$

where $\vec{v}_c$ is the velocity of C.M. frame, relative to the K-frame. Consequently, we may write,

$$\vec{M} = \sum m_i \left(\vec{r}_i \times \vec{v}_i\right) = \sum m_i \left(\vec{r}_i\,' \times \vec{\tilde{v}}_i\right) + \sum m_i \left(\vec{r}_i \times \vec{v}_c\right)$$

or, $$\vec{M} = \tilde{M} + m\left(\vec{r}_c \times \vec{v}_c\right), \text{ as } \sum m_i \vec{r}_i = m\vec{r}_c, \text{ where } m = \sum m_i.$$

or, $$\vec{M} = \tilde{M} + (\vec{r}_c \times m\,\vec{v}_c) = \tilde{M} + (\vec{r}_c \times \vec{p})$$

1.198 From conservation of linear momentum along the direction of incident ball for the system consists with colliding ball and phhere

$$mv_0 = mv' + \frac{m}{2} v_1 \qquad (1)$$

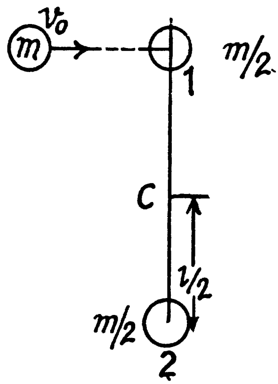
where v' and v_1 are the velocities of ball and sphere 1 respectively after collision. (Remember that the collision is head on).

As the collision is perfectly elastic, from the definition of co-efficeint of restitution,

$$1 = \frac{v' - v_1}{0 - v_0} \quad \text{or,} \quad v' - v_1 = -v_0 \qquad (2)$$

Solving (1) and (2), we get,

$v_1 = \frac{4\,v_0}{3}$, directed towards right.



In the C.M. frame of spheres 1 and 2 (Fig.)

$\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$ and $|\vec{\tilde{p}}_1| = |\vec{\tilde{p}}_2| = \mu\,|\vec{v}_1 - \vec{v}_2|$

Also, $\vec{r}_{1C} = -\vec{r}_{2C}$, thus $\vec{\tilde{M}} = 2\,[\vec{r}_{1C} \times \vec{\tilde{p}}_1]$

As $\vec{r}_{1C} \perp \vec{\tilde{p}}_1$, so, $\vec{\tilde{M}} = 2\left[\frac{l}{2}\,\frac{m/2}{2}\,\frac{4\,v_0}{3}\,\hat{n}\right]$

(where $\hat{n}$ is the unit vector in the sense of $\vec{r}_{1C} \times \vec{\tilde{p}}_1$)

Hence $\tilde{M} = \frac{m\,v_0\,l}{3}$

1.199 In the C.M. frame of the system (both the discs + spring), the linear momentum of the discs are related by the relation, $\vec{\tilde{p}}_1 = -\vec{\tilde{p}}_2$, at all the moments of time.

where, $\quad \tilde{p}_1 = \tilde{p}_2 = \tilde{p} = \mu\,v_{rel}$

And the total kinetic energy of the system,

$$T = \frac{1}{2}\mu\,v_{rel}^2 \text{ [See solution of 1.147 (b)]}$$

Bearing in mind that at the moment of maximum deformation of the spring, the projection of $\vec{v}_{rel}$ along the length of the spring becomes zero, i.e. $v_{rel\,(x)} = 0$.

The conservation of mechanical energy of the considered system in the C.M. frame gives.

$$\frac{1}{2}\left(\frac{m}{2}\right)v_0^2 = \frac{1}{2}\kappa\,x^2 + \frac{1}{2}\left(\frac{m}{2}\right)v_{rel\,(y)}^2 \qquad (1)$$

Now from the conservation of angular momentum of the system about the C.M.,

$$\frac{1}{2}\left(\frac{l_0}{2}\right)\left(\frac{m}{2}v_0\right) = 2\left(\frac{l_0 + x}{2}\right)\frac{m}{2}v_{rel\,(y)}$$

or, $$v_{rel\,(y)} = \frac{v_0\,l_0}{(l_0 + x)} = v_0\left(1 + \frac{x}{l_0}\right)^{-1} \approx v_0\left(1 - \frac{x}{l_0}\right), \text{ as } x << l_0 \qquad (2)$$

Using (2) in (1), $$\frac{1}{2}m\,v_0^2\left[1 - \left(1 - \frac{x}{l_0}\right)^2\right] = \kappa\,x^2$$

or, $$\frac{1}{2}m\,v_0^2\left[1 - \left(1 - \frac{2x}{l_0} + \frac{x^2}{l_0^2}\right)^2\right] = \kappa\,x^2$$

or, $$\frac{m\,v_0^2\,x}{l_0} \approx \kappa\,x^2, \text{ [neglecting } x^2/l_0^2\text{]}$$

As $$x \neq 0, \text{ thus } x = \frac{m\,v_0^2}{\kappa\,l_0}$$

1.4 UNIVERSAL GRAVITATION

1.200 We have

$$\frac{Mv^2}{r} = \frac{\gamma M m_s}{r^2} \text{ or } r = \frac{\gamma m_s}{v^2}$$

Thus

$$\omega = \frac{v}{r} = \frac{v}{\gamma m_s / v^2} = \frac{v^3}{\gamma m_s}$$

(Here m_s is the mass of the Sun.)

So $\quad T = \frac{2\pi \gamma m_s}{v^3} = \frac{2\pi \times 6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(34{\cdot}9 \times 10^3)^3} = 1{\cdot}94 \times 10^7 \text{ sec} = 225 \text{ days.}$

(The answer is incorrectly written in terms of the planetary mass M)

1.201 For any planet

$$M R \omega^2 = \frac{\gamma M m_s}{R^2} \text{ or } \omega = \sqrt{\frac{\gamma m_s}{R^3}}$$

So,

$$T = \frac{2\pi}{\omega} = 2\pi R^{3/2} / \sqrt{\gamma m_s}$$

(a) Thus

$$\frac{T_J}{T_E} = \left(\frac{R_J}{R_E}\right)^{3/2}$$

So

$$\frac{R_J}{R_E} = (T_J / T_E)^{2/3} = (12)^{2/3} = 5 \cdot 24.$$

(b)

$$V_J^2 = \frac{\gamma m_s}{R_J}, \text{ and } R_J = \left(T \frac{\sqrt{\gamma m_s}}{2\pi}\right)^{2/3}$$

So

$$V_J^2 = \frac{(\gamma m_s)^{2/3} (2\pi)^{2/3}}{T^{2/3}} \quad \text{or, } V_J = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3}$$

where $T = 12$ years. $m_s =$ mass of ths Sun.

Putting the values we get $V_J = 12{\cdot}97$ km/s

$$\text{Acceleration} = \frac{v_J^2}{R_J} = \left(\frac{2\pi \gamma m_s}{T}\right)^{2/3} \times \left(\frac{2\pi}{T\sqrt{\gamma m_s}}\right)^{2/3}$$

$$= \left(\frac{2\pi}{T}\right)^{4/3} (\gamma m_s)^{1/3}$$

$$= 2{\cdot}15 \times 10^{-4} \text{ km/s}^2$$

1.202 Semi-major axis= $(r+R)/2$

It is sufficient to consider the motion be along a circle of semi-major axis $\frac{r+R}{2}$ for T does not depend on eccentricity.

Hence $$T = \frac{2\pi\left(\frac{r+R}{2}\right)^{3/2}}{\sqrt{\gamma m_s}} = \pi\sqrt{(r+R)^3/2\gamma m_s}$$

(again m_s is the mass of the Sun)

1.203 We can think of the body as moving in a very elongated orbit of maximum distance R and minimum distance 0 so semi major axis $= R/2$. Hence if τ is the time of fall then

$$\left(\frac{2\tau}{T}\right)^2 = \left(\frac{R/2}{R}\right)^3 \quad \text{or} \quad \tau^2 = T^2/32$$

or $$\tau = T/4\sqrt{2} = 365/4\sqrt{2} = 64{\cdot}5 \text{ days.}$$

1.204 $T = 2\pi R^{3/2}/\sqrt{\gamma m_s}$

If the distances are scaled down, $R^{3/2}$ decreases by a factor $\eta^{3/2}$ and so does m_s. Hence T does not change.

1.205 The double star can be replaced by a single star of mass $\frac{m_1 m_2}{m_1 + m_2}$ moving about the centre of mass subjected to the force $\gamma m_1 m_2 / r^2$. Then

$$T = \frac{2\pi r^{3/2}}{\sqrt{\gamma m_1 m_2 \Big/ \frac{m_1 m_2}{m_1+m_2}}} = \frac{2\pi r^{3/2}}{\sqrt{\gamma M}}$$

So $$r^{3/2} = \frac{T}{2\pi}\sqrt{\gamma M}$$

or, $$r = \left(\frac{T}{2\pi}\right)^{2/3}(\gamma M)^{1/3} = \sqrt[3]{\gamma M (T/2\pi)^2}$$

1.206 (a) The gravitational potential due to m_1 at the point of location of m_2 :

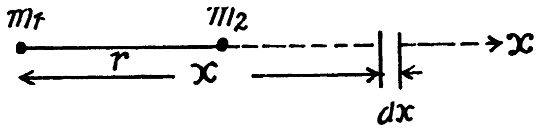
$$V_2 = \int_r^\infty \vec{G}\cdot d\vec{r} = \int_r^\infty -\frac{\gamma m_1}{x^2}\,dx = -\frac{\gamma m_1}{r}$$

So, $$U_{21} = m_2 V_2 = -\frac{\gamma m_1 m_2}{r}$$

Similarly $$U_{12} = -\frac{\gamma m_1 m_2}{r}$$

Hence

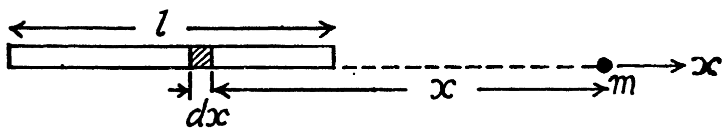
$$U_{12} = U_{21} = U = -\frac{\gamma m_1 m_2}{r}$$



(b) Choose the location of the point mass as the origin. Then the potential energy dU of an element of mass $dM = \frac{M}{l}dx$ of the rod in the field of the point mass is

$$dU = -\gamma m \frac{M}{l} dx \frac{1}{x}$$

where x is the distance between the element and the point. (Note that the rod and the point mass are on a straight line.) If then a is the distance of the nearer end of the rod from the point mass.



$$U = -\gamma \frac{mM}{l} \int_a^{a+l} \frac{dx}{x} = -\gamma m \frac{M}{l} \ln\left(1 + \frac{l}{a}\right)$$

The force of interaction is

$$F = -\frac{\partial U}{\partial a}$$

$$= \gamma \frac{mM}{l} \times \frac{1}{1 + \frac{l}{a}}\left(-\frac{l}{a^2}\right) = -\frac{\gamma m M}{a(a+l)}$$

Minus sign means attraction.

1.207 As the planet is under central force (gravitational interaction), its angular momentum is conserved about the Sun (which is situated at one of the focii of the ellipse)

So, $$m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1^2 = \frac{v_2^2 r_2^2}{r_1^2} \tag{1}$$

From the conservation of mechanical energy of the system (Sun + planet),

$$-\frac{\gamma m_s m}{r_1} + \frac{1}{2} m v_1^2 = -\frac{\gamma m_s m}{r_2} + \frac{1}{2} m v_2^2$$

or, $$-\frac{\gamma m_s}{r_1} + \frac{1}{2} v_2^2 \frac{r_2^2}{r_1^2} = -\left(\frac{\gamma m_s}{r_2}\right) + \frac{1}{2} v_2^2 \quad \text{[Using (1)]}$$

Thus, $$v_2 = \sqrt{2 \gamma m_s r_1 / r_2 (r_1 + r_2)} \tag{2}$$

Hence $$M = m v_2 r_2 = m \sqrt{2 \gamma m_s r_1 r_2 / (r_1 + r_2)}$$

1.208 From the previous problem, if r_1 , r_2 are the maximum and minimum distances from the sun to the planet and v_1 , v_2 are the corresponding velocities, then, say,

$$E = \frac{1}{2} m v_2^2 - \frac{\gamma m m_s}{r_2}$$

$$= \frac{\gamma m m_s}{r_1 + r_2} \cdot \frac{r_1}{r_2} - \frac{\gamma m m_s}{r_2} = -\frac{\gamma m m_s}{r_1 + r_2} = -\frac{\gamma m m_s}{2a} \text{ [Using Eq. (2) of 1.207]}$$

where $2a$ = major axis $= r_1 + r_2$. The same result can also be obtained directly by writing an equation analogous to Eq (1) of problem 1.191.

$$E = \frac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} - \frac{\gamma m m_s}{r}$$

(Here M is angular momentum of the planet and m is its mass). For extreme position $\dot{r} = 0$ and we get the quadratic

$$Er^2 + \gamma m m_s r - \frac{M^2}{2m} = 0$$

The sum of the two roots of this equation are

$$r_1 + r_2 = -\frac{\gamma m m_s}{E} = 2a$$

Thus $$E = -\frac{\gamma m m_s}{2a} = \text{ constant}$$

1.209 From the conservtion of angular momentum about the Sun.

$$m v_0 r_0 \sin\alpha = m v_1 r_1 = m v_2 r_2 \quad \text{or,} \quad v_1 r_1 = v_2 r_2 = v_0 r_2 \sin\alpha \tag{1}$$

From conservation of mechanical energy,

$$\frac{1}{2} m v_0^2 - \frac{\gamma m_s m}{r_0} = \frac{1}{2} m v_1^2 - \frac{\gamma m_s m}{r_1}$$

or, $$\frac{v_0^2}{2} - \frac{\gamma m_s}{r_0} = \frac{v_0^2 r_0^2 \sin^2\alpha}{2 r_1^2} - \frac{\gamma m_s}{r_1} \text{ (Using 1)}$$

or, $$\left(v_0^2 - \frac{2\gamma m_s}{r_0}\right) r_1^2 + 2\gamma m_s r_1 - v_0^2 r_0^2 \sin\alpha = 0$$

So, $$r_1 = \frac{-2\gamma m_s \pm \sqrt{4\gamma^2 m_s^2 + 4\left(v_0^2 r_0^2 \sin^2\alpha\right)\left(v_0^2 - \frac{2\gamma m_s}{r_0}\right)}}{2\left(v_0^2 - \frac{2\gamma m_s}{r_0}\right)}$$

$$= \frac{1 \pm \sqrt{1 - \frac{v_0^2 r_0^2 \sin^2\alpha}{\gamma m_s}\left(\frac{2}{r_0} - \frac{v_0^2}{r m_s}\right)}}{\left(\frac{2}{r_0} - \frac{v_0}{\gamma m_s}\right)} = \frac{r_0\left[1 \pm \sqrt{1 - (2-\eta)\eta \sin^2\alpha}\,\right]}{(2-\eta)}$$

where $\eta = v_0^2 r_0 / \gamma m_s$, ($m_s$ is the mass of the Sun).

1.210 At the minimum separation with the Sun, the cosmic body's velocity is perpendicular to its position vector relative to the Sun. If r_{min} be the sought minimum distance, from conservation of angular momentum about the Sun (C).

$$m v_0 l = m v r_{min} \text{ or, } v = \frac{v_0 l}{r_{min}} \qquad (1)$$

From conservation of mechanical energy of the system (sun + cosmic body),

$$\frac{1}{2} m v_0^2 = -\frac{\gamma m_s m}{r_{min}} + \frac{1}{2} m v^2$$

So, $$\frac{v_0^2}{2} = -\frac{\gamma m_s}{r_{min}} + \frac{v_0^2}{2 r_{min}^2} \quad \text{(using 1)}$$

or, $$v_0^2 r_{min}^2 + 2\gamma m_s r_{min} - v_0^2 l^2 = 0$$

So, $$r_{min} = \frac{-2\gamma m_s \pm \sqrt{4\gamma^2 m^2 + 4 v_0^2 v_0^2 l^2}}{2 v_0^2} = \frac{-\gamma m_s \pm \sqrt{\gamma^2 m_s^2 + v_0^4 l^2}}{v_0^2}$$

Hence, taking positive root

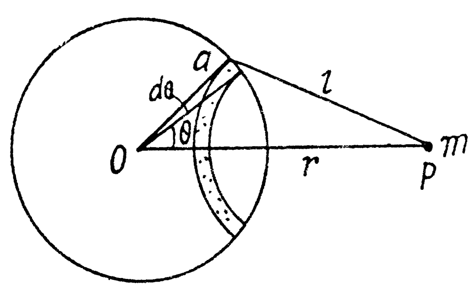
$$r_{min} = (\gamma m_s / v_0^2)\left[\sqrt{1 + (l v_0^2 / \gamma m_s)^2} - 1\right]$$

1.211 Suppose that the sphere has a radius equal to a. We may imagine that the sphere is made up of concentric thin spherical shells (layers) with radii ranging from 0 to a, and each spherical layer is made up of elementry bands (rings). Let us first calculate potential due to an elementry band of a spherical layer at the point of location of the point mass m (say point P) (Fig.). As all the points of the band are located at the distance l from the point P, so,

$$\partial \varphi = -\frac{\gamma\, \partial M}{l} \text{ (where mass of the band)} \qquad (1)$$

$$\partial M = \left(\frac{dM}{4\pi a^2}\right)(2\pi a \sin\theta)(a\, d\theta)$$

$$= \left(\frac{dM}{2}\right)\sin\theta\, d\theta \qquad (2)$$



And $$l^2 = a^2 + r^2 - 2 ar\cos\theta \qquad (3)$$

Differentiating Eq. (3), we get

$$l\,dl = ar \sin\theta\, d\theta \qquad (4)$$

Hence using above equations

$$\partial \varphi = -\left(\frac{\gamma\, dM}{2ar}\right) dl \qquad (5)$$

Now integrating this Eq. over the whole spherical layer

$$d\varphi = \int \partial \varphi = -\frac{\gamma\, dM}{2ar} \int_{r-a}^{r+a}$$

So $$d\varphi = -\frac{\gamma\, dM}{r} \tag{6}$$

Equation (6) demonstrates that the potential produced by a thin uniform spherical layer outside the layer is such as if the whole mass of the layer were concentrated at it's centre; Hence the potential due to the sphere at point P;

$$\varphi = \int d\varphi = -\frac{\gamma}{r}\int dM = -\frac{\gamma M}{r} \tag{7}$$

This expression is similar to that of Eq. (6)

Hence thte sought potential energy of gravitational interaction of the particle m and the sphere,

$$U = m\varphi = -\frac{\gamma M m}{r}$$

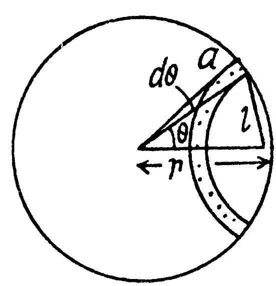
(b) Using the Eq., $$G_r = -\frac{\partial \varphi}{\partial r}$$

$$G_r = -\frac{\gamma M}{r^2} \qquad \text{(using Eq. 7)}$$

So $$\vec{G} = -\frac{\gamma M}{r^3}\vec{r} \text{ and } \vec{F} = m\vec{G} = -\frac{\gamma m M}{r^3}\vec{r} \tag{8}$$

1.212 (The problem has already a clear hint in the answer sheet of the problem book). Here we adopt a different method.

Let m be the mass of the spherical layer, wich is imagined to be made up of rings. At a point inside the spherical layer at distance r from the centre, the gravitational potential due to a ring element of radius a equals,



$d\varphi = -\frac{\gamma m}{2ar}\, dl$ (see Eq. (5) of solution of 1.211)

So, $$\varphi = \int d\varphi = -\frac{\gamma m}{2ar}\int_{a-r}^{a+r} dl = -\frac{\gamma m}{a} \tag{1}$$

Hence $$G_r = -\frac{\partial \varphi}{\partial r} = 0 \,.$$

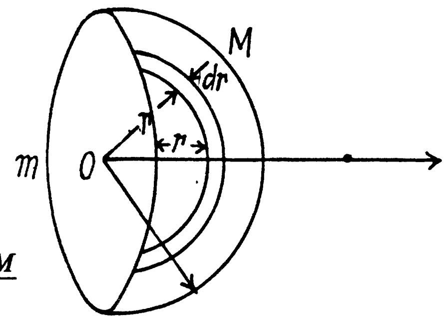
Hence gravitational field strength as well as field force becomes zero, inside a thin sphereical layer.

1.213 One can imagine that the uniform hemisphere is made up of thin hemispherical layers of radii ranging from 0 to R. Let us consider such a layer (Fig.). Potential at point O, due to this layer is,

$$d\varphi = -\frac{\gamma dm}{r} = -\frac{3\gamma M}{R^3} r dr, \text{ where } dm = \frac{M}{(2/3)\pi R^3}\left(\frac{4\pi r^2}{2}\right) dr$$

(This is because all points of each hemispherical shell are equidistant from O.)

Hence, $\varphi = \int d\varphi = -\frac{3\gamma M}{R^3}\int_0^R r dr = -\frac{3\gamma M}{2R}$



Hence, the work done by the gravitational field force on the particle of mass m, to remove it to infinity is given by the formula

$A = m\varphi$, since $\varphi = 0$ at infinity.

Hence the sought work,

$$A_{0\to\infty} = -\frac{3\gamma mM}{2R}$$

(The work done by the external agent is $-A$.)

1.214 In the solution of problem 1.211, we have obtained φ and G due to a uniform shpere, at a distance r from it's centre outside it. We have from Eqs. (7) and (8) of 1.211,

$$\varphi = -\frac{\gamma M}{r} \text{ and } \vec{G} = -\frac{\gamma M}{r^3}\vec{r} \quad \text{(A)}$$

Accordance with the Eq. (1) of the solution of 1.212, potential due to a spherical shell of radius a, at any point, inside it becomes

$$\varphi = \frac{\gamma M}{a} = \text{Const. and } G_r = -\frac{\partial \varphi}{\partial r} = 0 \quad \text{(B)}$$

For a point (say P) which lies inside the uniform solid sphere, the potential φ at that point may be represented as a sum.

$$\varphi_{inside} = \varphi_1 + \varphi_2$$

where φ_1 is the potential of a solid sphere having radius r and φ_2 is the potential of the layer of radii r and R. In accordance with equation (A)

$$\varphi_1 = -\frac{\gamma}{r}\left(\frac{M}{(4/3)\pi R^3}\frac{4}{3}\pi r^3\right) = -\frac{\gamma M}{R^3} r^2$$

The potential φ_2 produced by the layer (thick shell) is the same at all points inside it. The potential φ_2 is easiest to calculate, for the point positioned at the layer's centre. Using Eq. (B)

$$\varphi_2 = -\gamma\int_r^R \frac{dM}{r} = -\frac{3}{2}\frac{\gamma M}{R^3}(R^2 - r^2)$$

where $dM = \frac{M}{(4/3)\pi R^3} 4\pi r^2 dr = \left(\frac{3M}{R^3}\right) r^2 dr$

is the mass of a thin layer between the radii r and $r + dr$.

Thus $$\varphi_{inside} = \varphi_1 + \varphi_2 = \left(\frac{\gamma M}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \quad \text{(C)}$$

From the Eq. $$G_r = \frac{-\partial \varphi}{\partial r}$$

$$G_r = \frac{\gamma M r}{R^3}$$

or $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma \frac{4}{3}\pi \rho \vec{r}$$

(where $\rho = \dfrac{M}{\frac{4}{3}\pi R^3}$, is the density of the sphere) (D)

The plots $\varphi(r)$ and $G(r)$ for a uniform sphere of radius R are shown in figure of answersheet.

Alternate : Like Gauss's theorem of electrostatics, one can derive Gauss's theorem for gravitation in the form $\oint \vec{G} \cdot d\vec{S} = -4\pi\gamma m_{inclosed}$. For calculation of $\vec{G}$ at a point inside the sphere at a distance r from its centre, let us consider a Gaussian surface of radius r, Then,

$$G_r 4\pi r^2 = -4\pi\gamma\left(\frac{M}{R^3}\right)r^3 \quad \text{or,} \quad G_r = -\frac{\gamma M}{R^3}r$$

Hence, $$\vec{G} = -\frac{\gamma M}{R^3}\vec{r} = -\gamma\frac{4}{3}\pi\rho\vec{r}\left(\text{as } \rho = \frac{M}{(4/3)\pi R^3}\right)$$

So, $$\varphi = \int_r^\infty G_r\, dr = \int_r^R -\frac{\gamma M}{R^3}r\, dr + \int_R^\infty -\frac{\gamma M}{r^2}dr$$

Integrating and summing up, we get,

$$\varphi = -\frac{\gamma M}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

And from Gauss's theorem for outside it :

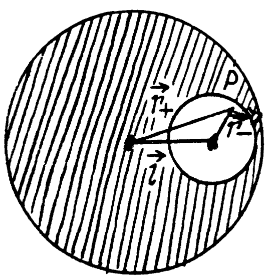
$$G_r 4\pi r^2 = -4\pi\gamma M \quad \text{or} \quad G_r = -\frac{\gamma M}{r^2}$$

Thus $$\varphi(r) = \int_r^\infty G_r\, dr = -\frac{\gamma M}{r}$$

1.215 Treating the cavity as negative mass of density $-\rho$ in a uniform sphere density $+\rho$ and using the superposition principle, the sought field strength is :

$$\vec{G} = \vec{G}_1 + \vec{G}_2$$

or $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\vec{r}_+ + -\frac{4}{3}\gamma\pi(-\rho)\vec{r}_-$$



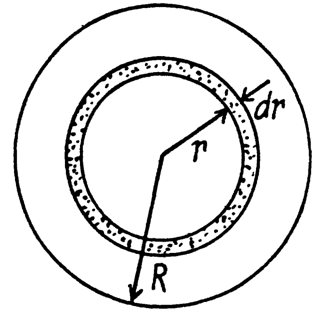
(where $\vec{r}_+$ and $\vec{r}_-$ are the position vectors of an orbitrary point P inside the cavity with respect to centre of sphere and cavity respectively.)

Thus $$\vec{G} = -\frac{4}{3}\pi\gamma\rho\left(\vec{r}_+ - \vec{r}_-\right) = -\frac{4}{3}\pi\gamma\rho\vec{l}$$

1.216 We partition the solid sphere into thin spherical layers and consider a layer of thickness dr lying at a distance r from the centre of the ball. Each spherical layer presses on the layers within it. The considered layer is attracted to the part of the sphere lying within it (the outer part does not act on the layer). Hence for the considered layer

$$dp\,4\pi r^2 = dF$$

or, $$dP\,4\pi r^2 = \frac{\gamma\left(\frac{4}{3}\pi r^3 \rho\right)(4\pi r^2 dr\,\rho)}{r^2}$$



(where ρ is the mean density of sphere)

or, $$dp = \frac{4}{3}\pi\gamma\rho^2 r\,dr$$

Thus $$p = \int_r^R dp = \frac{2\pi}{3}\gamma\rho^2(R^2 - r^2)$$

(The pressure must vanish at $r = R$.)

or, $$p = \frac{3}{8}\left(1 - (r^2/R^2)\right)\gamma M^2/\pi R^4, \text{ Putting } \rho = M/(4/3)\pi R^3$$

Putting $r = 0$, we have the pressure at sphere's centre, and treating it as the Earth where mean density is equal to $\rho = 5{\cdot}5 \times 10^3\ \text{kg/m}^3$ and $R = 64 \times 10^2$ km

we have, $$p = 1{\cdot}73 \times 10^{11}\ \text{Pa or } 1{\cdot}72 \times 10^6\ \text{atms.}$$

1.217 (a) Since the potential at each point of a spherical surface (shell) is constant and is equal to $\varphi = -\frac{\gamma m}{R}$, [as we have in Eq. (1) of solution of problem 1.212]

We obtain in accordance with the equation

$$U = \frac{1}{2}\int dm\,\varphi = \frac{1}{2}\varphi\int dm$$

$$= \frac{1}{2}\left(-\frac{\gamma m}{R}\right)m = -\frac{\gamma m^2}{2R}$$

(The factor $\frac{1}{2}$ is needed otherwise contribution of different mass elements is counted twice.)

(b) In this case the potential inside the sphere depends only on r (see Eq. (C) of the solution of problem 1.214)

$$\varphi = -\frac{3\gamma m}{2R}\left(1 - \frac{r^2}{3R^2}\right)$$

Here dm is the mass of an elementry spherical layer confined between the radii r and $r + dr$:

$$dm = (4\pi r^2 dr\,\rho) = \left(\frac{3m}{R^3}\right)r^2 dr$$

$$U = \frac{1}{2}\int dm\, \varphi$$

$$= \frac{1}{2}\int_0^R \left(\frac{3\,m}{R^3}\right) r^2\, dr \left\{-\frac{3\,\gamma\, m}{2\,R}\left(1 - \frac{r^2}{3\,R^2}\right)\right\}$$

After integrating, we get

$$U = -\frac{3}{5}\frac{\gamma\, m^2}{R}$$

1.218 Let $\omega = \sqrt{\frac{\gamma M_E}{r^3}}$ = circular frequency of the satellite in the outer orbit,

$\omega_0 = \sqrt{\frac{\gamma M_E}{(r - \Delta r)^3}}$ = circular frequency of the satellite in the inner orbit.

So, relative angular velocity = $\omega_0 \pm \omega$ where – sign is to be taken when the satellites are moving in the same sense and + sign if they are moving in opposite sense.
Hence, time between closest approaches

$$= \frac{2\pi}{\omega_0 \pm \omega} = \frac{2\pi}{\sqrt{\gamma M_E} / r^{3/2}}\frac{1}{\frac{3\,\Delta r}{2\,r} + \delta} = \begin{cases} 4{\cdot}5 \text{ days } (\delta = 0) \\ 0{\cdot}80 \text{ hour } (\delta = 2) \end{cases}$$

where δ is 0 in the first case and 2 in the second case.

1.219 $\omega_1 = \frac{\gamma M}{R^2} = \frac{6{\cdot}67 \times 10^{-11} \times 5{\cdot}96 \times 10^{24}}{(6{\cdot}37 \times 10^6)^2} = 9{\cdot}8 \text{ m/s}^2$

$$\omega_2 = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \left(\frac{2 \times 22}{24 \times 3600 \times 7}\right)^2 6{\cdot}37 \times 10^6 = 0{\cdot}034 \text{ m/s}^2$$

and $\quad \omega_3 = \frac{\gamma M_S}{R^2_{mean}} = \frac{6{\cdot}67 \times 10^{-11} \times 1{\cdot}97 \times 10^{30}}{(149{\cdot}50 \times 10^6 \times 10^3)^2} = 5{\cdot}9 \times 10^{-3} \text{ m/s}^2$

Then $\quad \omega_1 : \omega_2 : \omega_3 = 1 : 0{\cdot}0034 : 0{\cdot}0006$

1.220 Let h be the sought height in the first case. so

$$\frac{99}{100}g = \frac{\gamma M}{(R + h)^2}$$

$$= \frac{\gamma M}{R^2\left(1 + \frac{h}{R}\right)^2} = \frac{g}{\left(1 + \frac{h}{R}\right)^2}$$

or $$\frac{99}{100} = \left(1 + \frac{h}{R}\right)^{-2}$$

From the statement of the problem, it is obvious that in this case $h << R$

Thus $$\frac{99}{100} = \left(1 - \frac{2h}{R}\right) \text{ or } h = \frac{R}{200} = \left(\frac{6400}{200}\right) \text{km} = 32 \text{ km}$$

In the other case if h' be the sought height, than

$$\frac{g}{2} = g\left(1 + \frac{h'}{R}\right)^{-2} \text{ or } \frac{1}{2} = \left(1 + \frac{h'}{R}\right)^{-2}$$

From the language of the problem, in this case h' is not very small in comparision with R. Therefore in this case we cannot use the approximation adopted in the previous case.

Here, $\left(1 + \frac{h'}{R}\right)^2 = 2$ So, $\frac{h'}{R} = \pm \sqrt{2} - 1$

As $-ve$ sign is not acçeptable

$$h' = (\sqrt{2} - 1) R = (\sqrt{2} - 1)\, 6400 \text{ km} = 2650 \text{ km}$$

1.221 Let the mass of the body be m and let it go upto a height h.
From conservation of mechanical energy of the system

$$-\frac{\gamma M m}{R} + \frac{1}{2} m v_0^2 = \frac{-\gamma M m}{(R + h)} + 0$$

Using $\frac{\gamma M}{R^2} = g$, in above equation and on solving we get,

$$h = \frac{R v_0^2}{2 g R - v_0^2}$$

1.222 Gravitational pull provides the required centripetal acceleration to the satelite. Thus if h be the sought distance, we have

so, $$\frac{m v^2}{(R + h)} = \frac{\gamma m M}{(R + h)^2} \text{ or, } (R + h) v^2 = \gamma M$$

or, $$R v^2 + h v^2 = g R^2, \text{ as } g = \frac{\gamma M}{R^2}$$

Hence $$h = \frac{g R^2 - R v^2}{v^2} = R\left[\frac{g R}{v^2} - 1\right]$$

1.223 A satellite that hovers above the earth's equator and corotates with it moving from the west to east with the diurnal angular velocity of the earth appears stationary to an observer on the earth. It is called geostationary. For this calculation we may neglect the annual motion of the earth as well as all other influences. Then, by Newton's law,

$$\frac{\gamma M m}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r$$

where M = mass of the earth, T = 86400 seconds = period of daily rotation of the earth and r = distance of the satellite from the centre of the earth. Then

$$r = \sqrt[3]{\gamma M \left(\frac{T}{2\pi}\right)^2}$$

Substitution of $M = 5{\cdot}96 \times 10^{24}$ kg gives

$$r = 4{\cdot}220 \times 10^4 \text{ km}$$

The instantaneous velocity with respect to an inertial frame fixed to the centre of the earth at that moment will be

$$\left(\frac{2\pi}{T}\right) r = 3{\cdot}07 \text{ km/s}$$

and the acceleration will be the centripetal acceleration.

$$\left(\frac{2\pi}{T}\right)^2 r = 0{\cdot}223 \text{ m/s}^2$$

1.224 We know from the previous problem that a satellite moving west to east at a distance $R = 2{\cdot}00 \times 10^4$ km from the centre of the earth will be revolving round the earth with an angular velocity faster than the earth's diurnal angualr velocity. Let

$$\omega = \text{angular velocity of the satellite}$$

$$\omega_0 = \frac{2\pi}{T} = \text{anuglar velocity of the earth. Then}$$

$$\omega - \omega_0 = \frac{2\pi}{\tau}$$

as the relative angular velocity with respect to earth. Now by Newton's law

$$\frac{\gamma M}{R^2} = \omega^2 R$$

So,

$$M = \frac{R^3}{\gamma}\left(\frac{2\pi}{\tau} + \frac{2\pi}{T}\right)^2$$

$$= \frac{4\pi^2 R^3}{\gamma T^2}\left(1 + \frac{T}{\tau}\right)^2$$

Substitution gives

$$M = 6{\cdot}27 \times 10^{24} \text{ kg}$$

1.225 The velocity of the satellite in the inertial space fixed frame is $\sqrt{\frac{\gamma M}{R}}$ east to west. With respect to the Earth fixed frame, from the $\vec{v}_1{}' = \vec{v} - (\vec{w} \times \vec{r})$ the velocity is

$$v' = \frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}} = 7{\cdot}03 \text{ km/s}$$

Here M is the mass of the earth and T is its period of rotation about its own axis.

It would be $-\frac{2\pi R}{T}+\sqrt{\frac{\gamma M}{R}}$, if the satellite were moving from west to east.

To find the acceleration we note the formula

$$m\vec{w}' = \vec{F} + 2m(\vec{v}' \times \vec{\omega}) + m\omega^2 \vec{R}$$

Here $\vec{F} = -\frac{\gamma M m}{R^3}\vec{R}$ and $\vec{v}' \perp \vec{\omega}$ and $\vec{v}' \times \vec{\omega}$ is directed towards the centre of the Earth.

Thus $$w' = \frac{\gamma M}{R^2} + 2\left(\frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}}\right)\frac{2\pi}{T} - \left(\frac{2\pi}{T}\right)^2 R$$

toward the earth's rotation axis

$$= \frac{\gamma M}{R^2} + \frac{2\pi}{T}\left[\frac{2\pi R}{T} + 2\sqrt{\frac{\gamma M}{R}}\right] = 4\cdot 94 \text{ m/s}^2 \text{ on substitution.}$$

1.226 From the well known relationship between the velocities of a particle w.r.t a space fixed frame (K) rotating frame (K') $\vec{v} = \vec{v}' + (\vec{w} \times \vec{r})$

$$v'_1 = v - \left(\frac{2\pi}{T}\right)R$$

Thus kinetic energy of the satellite in the earth's frame

$$T_1' = \frac{1}{2}m v'^2_1 = \frac{1}{2}m\left(v - \frac{2\pi R}{T}\right)^2$$

Obviously when the satellite moves in opposite sense comared to the rotation of the Earth its velocity relative to the same frame would be

$$v_2' = v + \left(\frac{2\pi}{T}\right)R$$

And kinetic energy

$$T_2' = \frac{1}{2}m v'^2_2 = \frac{1}{2}m\left(v + \frac{2\pi R}{T}\right)^2 \quad (2)$$

From (1) and (2)

$$T' = \frac{\left(v + \frac{2\pi R}{T}\right)^2}{\left(v - \frac{2\pi R}{T}\right)^2} \quad (3)$$

Now from Newton's second law

$$\frac{\gamma M m}{R^2} = \frac{m v^2}{R} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{R}} = \sqrt{gR} \quad (4)$$

Using (4) and (3)

$$\frac{T_2'}{T_1'} = \frac{\left(\sqrt{gR} + \frac{2\pi R}{T}\right)^2}{\left(\sqrt{gR} - \frac{2\pi R}{T}\right)^2} = 1\cdot 27 \quad \text{nearly (Using Appendices)}$$

1.227 For a satellite in a circular orbit about any massive body, the following relation holds between kinetic, potential & total energy :

$$T = -E,\ U = 2E \quad (1)$$

Thus since total mechanical energy must decrease due to resistance of the cosmic dust, the kintetic energy will increase and the satellite will 'fall', We see then, by work energy theorm

$$dT = -dE = -dA_{fr}$$

So, $$mv\,dv = \alpha v^2 v\,dt \quad \text{or,} \quad \frac{\alpha\,dt}{m} = \frac{dv}{v^2}$$

Now from Netow's law at an arbitray radius r from the moon's centre.

$$\frac{v^2}{r} = \frac{\gamma M}{r^2} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{r}}$$

(M is the mass of the moon.) Then

$$v_i = \sqrt{\frac{\gamma M}{\eta R}},\ v_f = \sqrt{\frac{\gamma M}{R}}$$

where R= moon's radius. So

$$\int_{v_1}^{v_f} \frac{dv}{v^2} = \frac{\alpha}{m}\int_0^{\tau} dt = \frac{\alpha\tau}{m}$$

or, $$\tau = \frac{m}{\alpha}\left(\frac{1}{v_i} - \frac{1}{v_f}\right) = \frac{m}{\alpha\sqrt{\frac{M}{\gamma R}}}(\sqrt{\eta} - 1) = \frac{m}{\alpha\sqrt{gR}}(\sqrt{\eta} - 1)$$

where g is moon's gravity. The averaging implied by Eq. (1) (for noncircular orbits) makes the result approximate.

1.228 From Newton's second law

$$\frac{\gamma Mm}{R^2} = \frac{mv_0^2}{R} \text{ or } v_0 = \sqrt{\frac{\gamma M}{R}} = 1{\cdot}67\ \text{km/s} \quad (1)$$

From conservation of mechanical energy

$$\frac{1}{2}mv_e^2 - \frac{\gamma Mm}{R} = 0 \text{ or, } v_e = \sqrt{\frac{2\gamma M}{R}} = 2{\cdot}37\ \text{km/s}^2 \quad (2)$$

In Eq. (1) and (2), M and R are the mass of the moon and its radius. In Eq. (1) if M and R represent the mass of the earth and its radius, then, using appendices, we can easily get

$$v_0 = 7{\cdot}9 \text{ km/s} \quad \text{and} \quad v_c = 11{\cdot}2 \text{ km/s}.$$

1.229 In a parabolic orbit, $E = 0$

So
$$\frac{1}{2} m v_i^2 - \frac{\gamma M m}{R} = 0 \text{ or, } v_i = \sqrt{2}\sqrt{\frac{\gamma M}{R}}$$

where M = mass of the Moon, R = its radius. (This is just the escape velocity.)
On the other hand in orbit

$$m v_f^2 R = \frac{\gamma M m}{R^2} \text{ or } v_f = \sqrt{\frac{\gamma M}{R}}$$

Thus
$$\Delta v = (1 - \sqrt{2}) \sqrt{\frac{\gamma M}{R}} = -0{\cdot}70 \text{ km/s}.$$

1.230 From 1.228 for the Earth surface

$$v_0 = \sqrt{\frac{\gamma M}{R}} \text{ and } v_e = \sqrt{\frac{2\gamma M}{R}}$$

Thus the sought additional velocity

$$\Delta v = v_e - v_0 = \sqrt{\frac{\gamma M}{R}} (\sqrt{2} - 1) = gR(\sqrt{2} - 1)$$

This 'kick' in velocity must be given along the direction of motion of the satellite in its orbit.

1.231 Let r be the sought distance, then

$$\frac{\gamma \eta M}{(nR - r)^2} = \frac{\gamma M}{r^2} \text{ or } \eta r^2 = (nR - r)^2$$

or
$$\sqrt{\eta}\, r = (nR - r) \text{ or } r = \frac{nR}{\sqrt{\eta} + 1} = 3{\cdot}8 \times 10^4 \text{ km}.$$

1.232 Between the earth and the moon, the potential energy of the spaceship will have a maximum at the point where the attractions of the earth and the moon balance each other. This maximum P.E. is approximately zero. We can also neglect the contribution of either body to the p.E. of the spaceship sufficiently near the other body. Then the minimum energy that must be imparted to the spaceship to cross the maximum of the P.E. is clearly (using E to denote the earth)

$$\frac{\gamma M_E m}{R_E}$$

With this energy the spaceship will cross over the hump in the P.E. and coast down the hill of p.E. towards the moon and crashland on it. What the problem seeks is the minimum energy reguired for softlanding. That reguies the use of rockets to loving about the braking of the spaceship and since the kinetic energy of the gases ejected from the rocket will always be positive, the total energy required for softlanding is greater than that required for crashlanding. To calculate this energy we assume that the rockets are used fairly close to the moon when the spaceship has nealy attained its terminal velocity on the moon $\sqrt{\frac{2\gamma M_O}{R_0}}$ where M_0 is the mass of the moon and R_0 is its radius. In general $dE = vdp$ and since the speed of the ejected gases is not less than the speed of the rocket, and momentum transfered to the ejected gases must equal the momentum of the spaceship the energy E of the gass ejected is not less than the kinetic energy of spaceship

$$\frac{\gamma M_0 m}{R_0}$$

Addding the two we get the minimum work done on the ejected gases to bring about the softlanding.

$$A_{min} \approx \gamma m \left(\frac{M_E}{R_E} + \frac{M_0}{R_0}\right)$$

On substitution we get $1{\cdot}3 \times 10^8$ kJ.

1.233 Assume first that the attraction of the earth can be neglected. Then the minimum velocity, that must be imparted to the body to escape from the Sun's pull, is, as in 1·230, equal to

$$(\sqrt{2} - 1)\, v_1$$

where $v_1^2 = \gamma M_s / r$, r = radius of the earth's orbit, M_s = mass of the Sun.

In the actual case near the earth, the pull of the Sun is small and does not change much over distances, which are several times the radius of the Earth. The velocity v_3 in question is that which overcomes the earth's pull with sufficient velocity to escape the Sun's pull. Thus

$$\frac{1}{2} m v_3^2 - \frac{\gamma M_E}{R} = \frac{1}{2} m (\sqrt{2} - 1)^2 v_1^2$$

where R = radius of the earth, M_E = mass of the earth.

Writing $v_1^2 = \gamma M_E / R$, we get

$$v_3 = \sqrt{2 v_2^2 + (\sqrt{2} - 1)^2 v_1^2} = 16{\cdot}6 \text{ km/s}$$

1.5 DYNAMICS OF A SOLID BODY

1.234 Since, motion of the rod is purely translational, net torque about the C.M. of the rod should be equal to zero.

Thus $$F_1 \frac{l}{2} = F_2 \left(\frac{l}{2} - a \right) \text{ or, } \frac{F_1}{F_2} = 1 - \frac{a}{l/2} \quad (1)$$

For the translational motion of rod.

$$F_2 - F_1 = m w_c \text{ or } 1 - \frac{F_1}{F_2} = \frac{m w_c}{F_2} \quad (2)$$

From (1) and (2)

$$\frac{a}{l/2} = \frac{m w_c}{F_2} \text{ or, } l = \frac{2 a F_2}{m w_c} = 1 \text{ m}$$

1.235 Sought moment $\vec{N} = \vec{r} \times \vec{F} = (a\vec{i} + b\vec{j}) \times (A\vec{i} + B\vec{j})$

$$= aB\vec{k} + Ab(-\vec{k}) = (aB - Ab)\vec{k}$$

and arm of the force $$l = \frac{N}{F} = \frac{aB - Ab}{\sqrt{A^2 + B^2}}$$

1.236 Relative to point O, the net moment of force :

$$\vec{N} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = (a\vec{i} \times A\vec{j}) + (B\vec{j} \times B\vec{i})$$

$$= ab\vec{k} + AB(-\vec{k}) = (ab - AB)\vec{k} \quad (1)$$

Resultant of the external force

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = A\vec{j} + B\vec{i} \quad (2)$$

As $\vec{N} \cdot \vec{F} = 0$ (as $\vec{N} \perp \vec{F}$) so the sought arm l of the force $\vec{F}$

$$l = N/F = \frac{ab - AB}{\sqrt{A^2 + B^2}}$$

1.237 For coplanar forces, about any point in the same plane, $\sum \vec{r}_i \times \vec{F}_i = \vec{r} \times \vec{F}_{net}$

(where $\vec{F}_{net} = \sum \vec{F}_i$ = resultant force) or, $\vec{N}_{net} = \vec{r} \times \vec{F}_{net}$

Thus length of the arm, $l = \dfrac{N_{net}}{F_{net}}$

Here obviously $|\vec{F}_{net}| = 2F$ and it is directed toward right along AC. Take the origin at C. Then about C,

$\vec{N} = \left(\sqrt{2}\, a F + \frac{a}{\sqrt{2}} F - \sqrt{2}\, a F \right)$ directed normally into the plane of figure.

(Here a = side of the square.)

Thus $\vec{N} = F \dfrac{a}{\sqrt{2}}$ directed into the plane of the figure.

Hence $$l = \frac{F(a/\sqrt{2})}{2F} = \frac{a}{2\sqrt{2}} = \frac{a}{2} \sin 45°$$

Thus the point of application of force is at the mid point of the side BC.

1.238 (a) Consider a strip of length dx at a perpendicular distance x from the axis about which we have to find the moment of inertia of the rod. The elemental mass of the rod equals

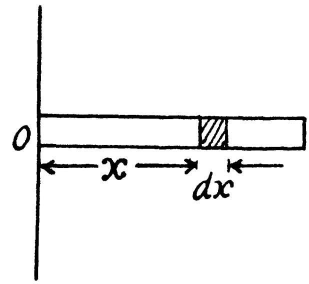
$$dm = \frac{m}{l} dx$$

Moment of inertia of this element about the axis

$$dI = dm\, x^2 = \frac{m}{l} dx \cdot x^2$$

Thus, moment of inertia of the rod, as a whole about the given axis

$$I = \int_0^l \frac{m}{l} x^2\, dx = \frac{m l^2}{3}$$



(b) Let us imagine the plane of plate as xy plane taking the origin at the intersection point of the sides of the plate (Fig.).

Obviously $$I_x = \int dm\, y^2$$

$$= \int_0^a \left(\frac{m}{ab} b\, dy \right) y^2$$

$$= \frac{m a^2}{3}$$

Similarly $$I_y = \frac{m b^2}{3}$$

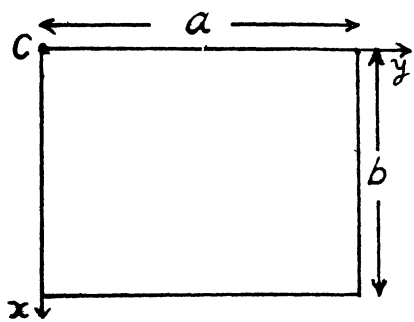
Hence from perpendicular axis theorem

$$I_z = I_x + I_y = \frac{m}{3}\left(a^2 + b^2\right),$$

which is the sought ,moment of inertia.

1.239 (a) Consider an elementry disc of thickness dx. Moment of inertia of this element about the z-axis, passing through its C.M.

$$dI_z = \frac{(dm) R^2}{2} = \rho S\, dx\, \frac{R^2}{2}$$



where ρ = density of the material of the plate and S = area of cross section of the plate.

Thus the sought moment of inertia

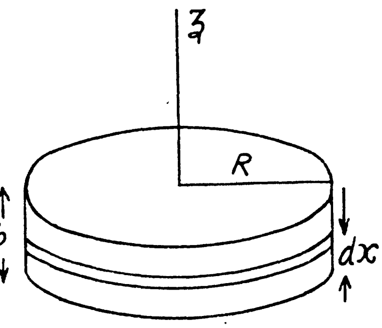
$$I_z = \frac{\rho S R^2}{2} \int_0^b dx = \frac{R^2}{2} \rho S b$$

$$= \frac{\pi}{2} \rho b R^4 \left(\text{as } S = \pi R^2\right)$$

putting all the vallues we get, $I_z = 2 \cdot \text{gm} \cdot \text{m}^2$

(b) Consider an element disc of radius r and thickness dx at a distance x from the point O. Then $r = x \tan\alpha$ and volume of the disc

$$= \pi x^2 \tan^2 \alpha \, dx$$



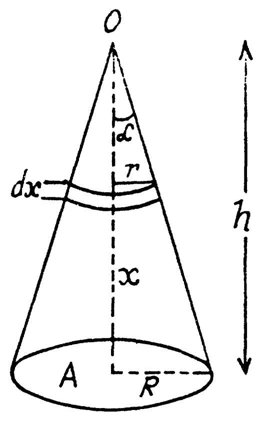
Hence, its mass $dm = \pi x^2 \tan\alpha . dx \cdot \rho$ (where ρ = density of the cone $= m / \frac{1}{3} \pi R^2 h$)

Moment of inertia of this element, about the axis OA,

$$dI = dm \frac{r^2}{2}$$

$$= (\pi x^2 \tan^2 \alpha \, dx) \frac{x^2 \tan^2 \alpha}{2}$$

$$= \frac{\pi \rho}{2} x^4 \tan^4 \alpha \, dx$$



Thus the sought moment of inertia $I = \frac{\pi \rho}{2} \tan^4 \alpha \int_0^h x^4 \, dx$

$$= \frac{\pi \rho R^4 \cdot h^5}{10 h^4} \left(\text{as } \tan\alpha = \frac{R}{h} \right)$$

Hence $\quad I = \frac{3 m R^2}{10} \left(\text{putting } \rho = \frac{3m}{\pi R^2 h} \right)$

1.240 (a) Let us consider a lamina of an arbitrary shape and indicate by 1,2 and 3, three axes coinciding with x, y and z – axes and the plane of lamina as $x - y$ plane.

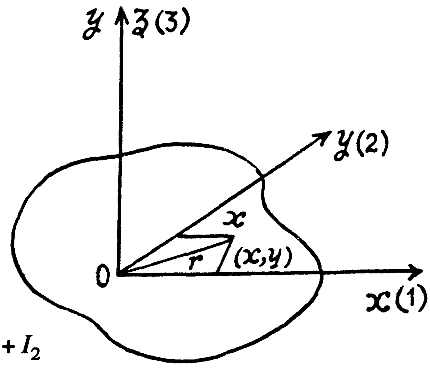
Now, moment of inertia of a point mass about x – axis, $dI_x = dm\, y^2$

Thus moment of inertia of the lamina about this axis, $I_x = \int dm y^2$

Similarly, $I_y = \int dm x^2$

and $I_z = \int dm r^2$

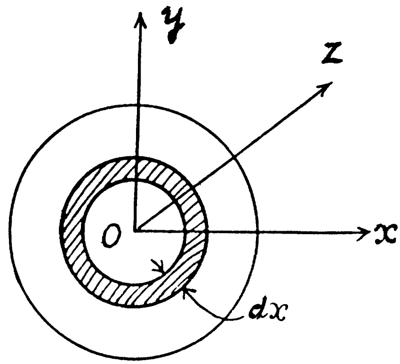
$= \int dm (x^2 + y^2)$ as $r = \sqrt{x^2 + y^2}$



Thus, $\quad I_z = I_x + I_y$ or, $I_3 = I_1 + I_2$

(b) Let us take the plane of the disc as $x - y$ plane and origin to the centre of the disc (Fig.) From the symmetry $I_x = I_y$. Let us consider a ring element of radius r and thickness dr, then the moment of inertia of the ring element about the y – axis.

$$dI_z = dm r^2 = \frac{m}{\pi R^2}(2\pi r dr) r^2$$



Thus the moment of inertia of the disc about z – axis

$$I_z = \frac{2m}{R^2}\int_0^R r^3\, dr = \frac{mR^2}{2}$$

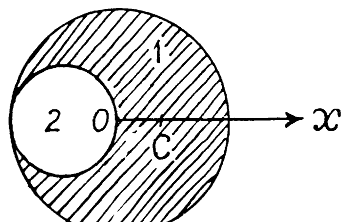
But we have $\quad I_z = I_x + I_y = 2I_x$

Thus $\quad I_x = \frac{I_z}{2} = \frac{mR^2}{4}$

1.241 For simplicity let us use a mathematical trick. We consider the portion of the given disc as the superposition of two complete discs (without holes), one of positive density and radius R and other of negative density but of same magnitude and radius $R/2$.

As (area) α (mass), the respective masses of the considered discs are $(4m/3)$ and $(-m/3)$ respectively, and these masses can be imagined to be situated at their respective centers (C.M). Let us take point O as origin and point x – axis towards right. Obviously the C.M. of the shaded position of given shape lies on the x – axis. Hence the C.M. (C) of the shaded portion is given by

$$x_c = \frac{(-m/3)(-R/2) + (4m/3)0}{(-m/3) + 4m/3} = \frac{R}{6}$$



Thus C.M. of the shape is at a distance $R/6$ from point O toward x – axis

Using parallel axis theorem and bearing in mind that the moment of inertia of a complete homogeneous disc of radius m_0 and radius r_0 equals $\frac{1}{2}m_0 r_0^2$. The moment of inetia of the small disc of mass $(-m/3)$ and radius $R/2$ about the axis passing through point C and perpendicular to the plane of the disc

$$I_{2C} = \frac{1}{2}\left(-\frac{m}{3}\right)\left(\frac{R}{2}\right)^2 + \left(-\frac{m}{3}\right)\left(\frac{R}{2}+\frac{R}{6}\right)^2$$

$$= -\frac{mR^2}{24} - \frac{4}{27}mR^2$$

Similarly $\quad I_{1C} = \frac{1}{2}\left(\frac{4m}{3}\right)R^2 + \left(\frac{4m}{3}\right)\left(\frac{R}{6}\right)^2$

$$= \frac{2}{3}mR^2 + \frac{mR^2}{27}$$

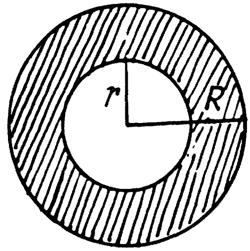
Thus the sought moment of inertia,

$$I_C = I_{1C} + I_{2C} = \frac{15}{24}mR^2 - \frac{3}{27}mR^2 = \frac{37}{72}mR^2$$

1.242 Moment of inertia of the shaded portion, about the axis passing through it's certre,

$$I = \frac{2}{5}\left(\frac{4}{3}\pi R^3\rho\right)R^2 - \frac{2}{5}\left(\frac{4}{3}\pi r^3\rho\right)r^3$$

$$= \frac{2}{5}\frac{4}{3}\pi\rho\left(R^5 - r^5\right)$$



Now, if $R = r + dr$, the shaded portion becomes a shell, which is the required shape to calculate the moment of inertia.

Now, $$I = \frac{2}{5} - \frac{4}{3}\pi\rho\left\{(r+dr)^5 - r^5\right\}$$

$$= \frac{2}{5}\frac{4}{3}\pi\rho\left(r^5 + 5r^4\,dr + \ldots\ldots - r^5\right)$$

Neglecting higher terms.

$$= \frac{2}{3}\left(4\pi r^2\,dr\rho\right)r^2 = \frac{2}{3}mr^2$$

1.243 (a) Net force which is effective on the system (cylinder M + body m) is the weight of the body m in a uniform gravitational field, which is a constant. Thus the initial acceleration of the body m is also constant.

From the conservation of mechanical energy of the said system in the uniform field of gravity at time $t = \Delta t$: $\Delta T + \Delta U = 0$

or $$\frac{1}{2}mv^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2 - mg\,\Delta h = 0$$

or, $$\frac{1}{4}(2m+M)v^2 - mg\,\Delta h = 0 \quad [\text{ as } v = \omega R \text{ at all times }] \qquad (1)$$

But $$v^2 = 2w\,\Delta h$$

Hence using it in Eq. (1), we get

$$\frac{1}{4}(2m+M)\,2w\,\Delta h - mg\,\Delta h = 0 \text{ or } w = \frac{2mg}{(2m+M)}$$

From the kinematical relationship, $\beta = \dfrac{w}{R} = \dfrac{2mg}{(2m+M)R}$

Thus the sought angular velocity of the cylinder

$$\omega(t) = \beta t = \frac{2mg}{(2m+M)R}t = \frac{gt}{(1+M/2m)R}$$

(b) Sought kinetic energy.

$$T(t) = \frac{1}{2}mv^2 + \frac{1}{2}\frac{Ml^2}{2}\omega^2 = \frac{1}{4}(2m+M)R^2\omega^2$$

1.244 For equilibrium of the disc and axle

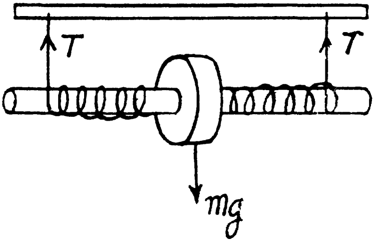
$$2T = mg \text{ or } T = mg/2$$

As the disc unwinds, it has an angular acceleration β given by

$$I\beta = 2Tr \text{ or } \beta = \frac{2Tr}{I} = \frac{mgr}{I}$$

The corresponding linear acceleration is

$$r\beta = w = \frac{mgr^2}{I}$$



Since the disc remains stationary under the combined action of this acceleration and the acceleration $(-w)$ of the bar which is transmitted to the axle, we must have

$$w = \frac{mgr^2}{I}$$

1.245 Let the rod be deviated through an angle $\varphi'$from its initial position at an arbitrary instant of time, measured relative to the initial position in the positive direction. From the equation of the increment of the mechanical energy of the system.

$$\Delta T = A_{ext}$$

or,
$$\frac{1}{2} I\omega^2 = \int N_z \, d\varphi$$

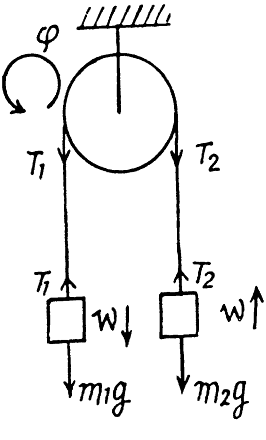
or,
$$\frac{1}{2}\frac{Ml^2}{3}\omega^2 = \int_0^{\varphi} Fl\cos\varphi \, d\varphi = Fl\sin\varphi$$

Thus,
$$\omega = \sqrt{\frac{6F\sin\varphi}{Ml}}$$

1.246 First of all, let us sketch free body diagram of each body. Since the cylinder is rotating and massive, the tension will be different in both the sections of threads. From Newton's law in projection form for the bodies m_1 and m_2 and noting that $w_1 = w_2 = w = \beta R$, (as no thread slipping), we have $(m_1 > m_2)$

$$m_1 g - T_1 = m_1 w = m_1 \beta R$$

$$\text{and } T_2 - m_2 g = m_2 w \qquad (1)$$



Now from the equation of rotational dynamics of a solid about stationary axis of rotation. i.e. $N_z = I\beta_z$, for the cylinder.

$$\text{or, } (T_1 - T_2)R = I\beta = mR^2\beta/2 \qquad (2)$$

Similtaneous solution of the above equations yields :

$$\beta = \frac{(m_1 - m_2)g}{R\left(m_1 + m_2 + \frac{m}{2}\right)} \text{ and } \frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$

1.247 As the system $(m + m_1 + m_2)$ is under constant forces, the acceleration of body m_1 an m_2 is constant. In addition to it the velocities and accelerations of bodies m_1 and m_2 aı equal in magnitude (say v and w) because the length of the thread is constant.
From the equation of increament of mechanical energy i.e. $\Delta T + \Delta U = A_{fr}$, at time t whe block m_1 is distance h below from initial position corresponding to $t = 0$,

$$\frac{1}{2}(m_1 + m_2) v^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\frac{v^2}{R^2} - m_2 gh = -km_1 gh \qquad (1$$

(as angular velocity $\omega = v/R$ for no slipping of thread.)

But $$v^2 = 2wh$$

So using it in (1), we get

$$w = \frac{2(m_2 - km_1) g}{m + 2(m_1 + m_2)}. \qquad (2$$

Thus the work done by the friction force on m_1

$$A_{fr} = -km_1gh = -km_1g\left(\frac{1}{2}wt^2\right)$$

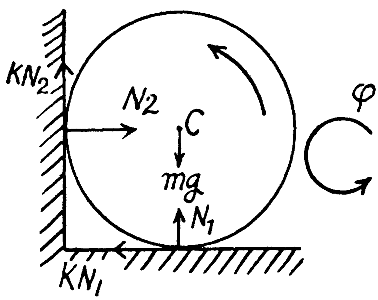
$$= -\frac{km_1(m_1 - km_1) g^2 t^2}{m + 2(m_1 + m_2)} \qquad \text{(using 2).}$$

1.248 In the problem, the rigid body is in translation equlibrium but there is an angular retardation. We first sketch the free body diagram of the cylinder. Obviously the friction forces, acting on the cylinder, are kinetic. From the condition of translational equlibrium for the cylinder,

$$mg = N_1 + kN_2; \;\; N_2 = kN_1$$

Hence, $$N_1 = \frac{mg}{1+k^2}; \; N_2 = k\frac{mg}{1+k^2}$$

For pure rotation of the cylinder about its rotation axis, $N_z = I\beta_z$



or, $$-kN_1R - kN_2R = \frac{mR^2}{2}\beta_z$$

or, $$-\frac{kmgR(1+k)}{1+k^2} = \frac{mR^2}{2}\beta_z$$

or, $$\beta_z = -\frac{2k(1+k)g}{(1+k^2)R}$$

Now, from the kinematical equation,

$$\omega^2 = \omega_0^2 + 2\beta_z \Delta\varphi \text{ we have,}$$

$$\Delta\varphi = \frac{\omega_0^2(1+k^2)R}{4k(1+k)g}, \qquad \text{because } \omega = 0$$

Hence, the sought number of turns,

$$n = \frac{\Delta\varphi}{2\pi} = \frac{\omega_0^2(1+k^2)R}{8\pi k(1+k)g}$$

1.249 It is the moment of friction force which brings the disc to rest. The force of friction is applied to each section of the disc, and since these sections lie at different distances from the axis, the moments of the forces of friction differ from section to section.

To find N_z, where z is the axis of rotation of the disc let us partition the disc into thin rings (Fig.). The force of friction acting on the considered element

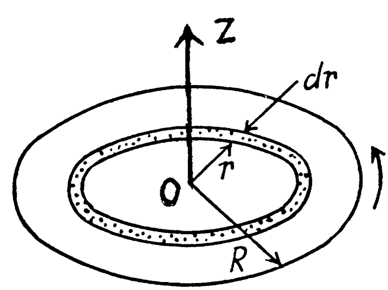
$dfr = k(2\pi r\, dr\, \sigma)g$, (where σ is the density of the disc)

The moment of this force of friction is

$$dN_z = -r\, dfr = -2\pi k\sigma g r^2 dr$$

Integrating with respect to r from zero to R, we get

$$N_z = -2\pi k\sigma g\int_0^R r^2 dr = -\frac{2}{3}\pi k\sigma g R^3.$$



For the rotation of the disc about the stationary axis z, from the equation $N_z = I\beta_z$

$$-\frac{2}{3}\pi k\sigma g R^3 = \frac{(\pi R^2\sigma)R^2}{2}\beta_z \quad \text{or} \quad \beta_z = -\frac{4kg}{3R}$$

Thus from the angular kinematical equation $\omega_z = \omega_{0z} + \beta_z t$

$$0 = \omega_0 + \left(-\frac{4kg}{3R}\right)t \quad \text{or} \quad t = \frac{3R\,\omega_0}{4kg}$$

1.250 According to the question,

$$I\frac{d\omega}{dt} = -k\sqrt{\omega} \quad \text{or,} \quad I\frac{d\omega}{\sqrt{\omega}} = -k\,dt$$

Integrating, $$\sqrt{\omega} = -\frac{kt}{2I} + \sqrt{\omega_0}$$

or, $$\omega = \frac{k^2t^2}{4I^2} - \frac{\sqrt{\omega_0}\,kt}{I} + \omega_0, \text{ (Noting that at } t = 0,\ \omega = \omega_0.)$$

Let the flywheel stops at $t = t_0$ then from Eq. (1), $t_0 = \dfrac{2I\sqrt{\omega_0}}{k}$

Hence sought average angular velocity

$$<\omega> = \frac{\displaystyle\int_0^{\frac{2I\sqrt{\omega_0}}{k}}\left(\frac{k^2t^2}{4I^2} - \frac{\sqrt{\omega_0}\,kt}{I} + \omega_0\right)dt}{\displaystyle\int_0^{\frac{2I\sqrt{\omega_0}}{k}} dt} = \frac{\omega_0}{3}$$

1.251 Let us use the equation $\frac{dM_z}{dt} = N_z$ relative to the axis through O (1)

For this purpose, let us find the angular momentum of the system M_z about the given rotation axis and the corresponding torque N_z. The angular momentum is

$$M_z = I\omega + mvR = \left(\frac{m_0}{2} + m\right) R^2 \omega$$

[where $I = \frac{m_0}{2} R^2$ and $v = \omega R$ (no cord slipping)]

So, $$\frac{dM_z}{dt} = \left(\frac{MR^2}{2} + mR^2\right)\beta_z \quad (2)$$

The downward pull of gravity on the overhanging part is the only external force, which exerts a torque about the z-axis, passing through O and is given by,

$$N_z = \left(\frac{m}{l}\right) xgR$$

Hence from the equation $$\frac{dM_z}{dt} = N_z$$

$$\left(\frac{MR^2}{2} + mR^2\right)\beta_z = \frac{m}{l} xgR$$

Thus, $$\beta_z = \frac{2mgx}{lR\,(M + 2m)} > 0$$

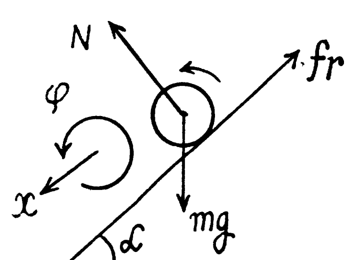
Note : *We may solve this problem using conservation of mechanical energy of the system (cylinder + thread) in the uniform field of gravity.*

1.252 (a) Let us indicate the forces acting on the sphere and their points of application. Choose positive direction of x and φ (rotation angle) along the incline in downward direction and in the sense of $\vec{\omega}$ (for undirectional rotation) respectively. Now from equations of dynamics of rigid body i.e. $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ we get :

$$mg \sin\alpha - f_r = mw \quad (1)$$

and $$fr\,R = \frac{2}{5} mR^2 \beta \quad (2)$$

But $$fr \le kmg \cos\alpha \quad (3)$$



In addition, the absence of slipping provides the kinematical realtionship between the accelerations :

$$w = \beta R \quad (4)$$

The simultaneous solution of all the four equations yields :

$$k \cos\alpha \ge \frac{2}{7} \sin\alpha, \quad \text{or} \quad k \ge \frac{2}{7} \tan\alpha$$

(b) Solving Eqs. (1) and (2) [of part (a)], we get :

$$w_c = \frac{5}{7} g \sin \alpha.$$

As the sphere starts at $t = 0$ along positive x axis, for pure rolling

$$v_c(t) = w_c t = \frac{5}{7} g \sin \alpha\, t \qquad (5)$$

Hence the sought kinetic energy

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2}\frac{2}{5} m R^2 \omega^2 = \frac{7}{10} m v_c^2 \text{ (as } \omega = v_c/R\text{)}$$

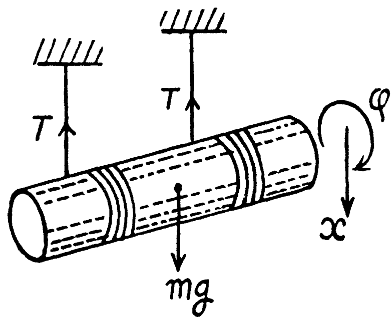
$$= \frac{7}{10} m \left(\frac{5}{7} g \sin \alpha\, t\right)^2 = \frac{5}{14} m g^2 \sin^2 \alpha\, t^2$$

1.253 (a) Let us indicate the forces and their points of application for the cylinder. Choosing the positive direction for x and φ as shown in the figure, we write the equation of motion of the cylinder axis and the equation of moments in the C.M. frame relative to that axis i.e. from equation $F_x = m w_c$ and $N_z = I_c \beta_z$

$$mg - 2T = m w_c; \; 2TR = \frac{mR^2}{2} \beta$$

As there is no slipping of thread on the cylinder

$$w_c = \beta R$$



From these three equations

$$T = \frac{mg}{6} = 13 \text{ N}, \; \beta = \frac{2}{5}\frac{g}{R} = 5 \times 10^2 \text{ rad/s}^2$$

(b) we have $\beta = \frac{2}{3}\frac{g}{R}$

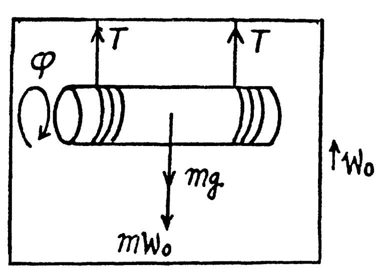
So, $w_c = \frac{2}{3} g > 0$ or, in vector form $\vec{w}_c = \frac{2}{3}\vec{g}$

$$P = \vec{F} \cdot \vec{v} = \vec{F} \cdot (\vec{w}_c t)$$

$$= m \vec{g} \cdot \left(\frac{2}{3} \vec{g} t\right) = \frac{2}{3} m g^2 t$$

1.254 Let us depict the forces and their points of application corresponding to the cylinder attached with the elevator. Newton's second law for solid in vector form in the frame of elevator, gives :

$$2\vec{T} + m\vec{g} + m(-\vec{w}_0) = m\vec{w}' \qquad (1)$$



The equation of moment in the C.M. frame relative to the cylinder axis i.e. from $N_z = I_c \beta_z$ ·

$$2TR = \frac{mR^2}{2} \beta = \frac{mR^2}{2} \frac{w'}{R}$$

[as thread does not slip on the cylinder, $w' = \beta R$]

or, $$T = \frac{mw'}{4}$$

As (1) $\vec{T} \uparrow\downarrow \vec{w}$

so in vector form

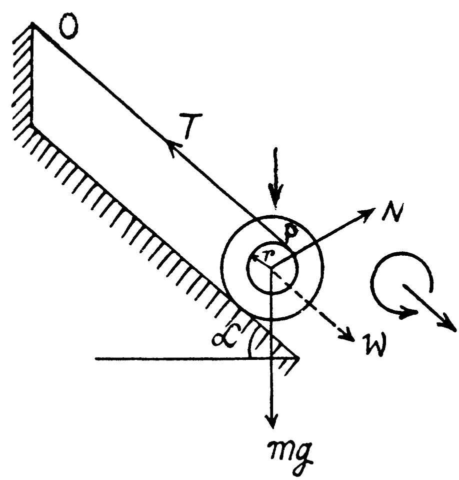
$$\vec{T} = -\frac{m\vec{w}}{4} \tag{2}$$

Solving Eqs. (1) and (2), $\vec{w}' = \frac{2}{3}(\vec{g} - \vec{w}_0)$ and sought force

$$\vec{F} = 2\vec{T} = \frac{1}{3} m (\vec{g} - \vec{w}_0).$$

1.255 Let us depict the forces and their points of application for the spool. Choosing the positive direction for x and φ as shown in the fig., we apply $F_x = mw_{cx}$ and $N_{cz} = I_c \beta_z$ and get

$$mg \sin\alpha - T = mw\,;\ Tr = I\beta$$



"Notice that if a point of a solid in plane motion is connected with a thread, the projection of velocity vector of the solid's point of contact along the length of the thread equals the velocity of the other end of the thread (if it is not slacked)"

Thus in our problem, $v_p = v_0$ but $v_0 = 0$, hence point P is the instantaneous centre of rotation of zero velocity for the spool. Therefore $v_c = \omega r$ and subsequently $w_c = \beta r$.

Solving the equations simultaneously, we get

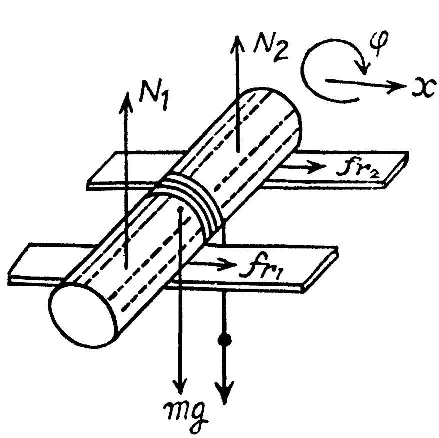
$$w = \frac{g \sin\alpha}{1 + \frac{I}{mr^2}} = 1{\cdot}6 \text{ m/s}^2$$

1.256 Let us sketch the force diagram for solid cylinder and apply Newton's second law in projection form along x and y axes (Fig.) :

$$fr_1 + fr_2 = mw_c \tag{1}$$

and $$N_1 + N_2 - mg - F = 0$$

or $$N_1 + N_2 = mg + F \tag{2}$$



Now choosing positive direction of φ as shown in the figure and using $N_{cz} = I_c \beta_z$, we get

$$FR - (fr_1 + fr_2)R = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w_c}{R} \tag{3}$$

[as for pure rolling $w_c = \beta R$]. In addition to,

$$fr_1 + fr_2 \leq k(N_1 + N_2) \tag{4}$$

Solving the Eqs., we get

$$F \le \frac{3\,k\,mg}{(2-3k)}, \quad \text{or} \quad F_{\max} = \frac{3\,k\,mg}{2-3\,k}$$

and

$$w_{c\,(\max)} = \frac{k\,(N_1+N_2)}{m}$$

$$= \frac{k}{m}\left[mg + F_{\max}\right] = \frac{k}{m}\left[mg + \frac{3\,kmg}{2-3k}\right] = \frac{2\,kg}{2-3k}$$

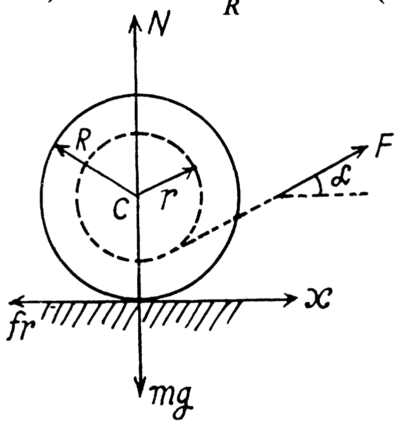
1.257 (a) Let us choose the positive direction of the rotation angle φ, such that w_{cx} and β_z have identical signs (Fig.). Equation of motion, $F_x = mw_{cx}$ and $N_{cz} = I_c\,\beta_z$ gives :

$$F\cos\alpha - fr = mw_{cx} : fr\,R - Fr = I_c\,\beta_z = \gamma m R^2\,\beta_z$$

In the absence of the slipping of the spool $w_{cx} = \beta_z R$

From the three equations $w_{cx} = w_c = \dfrac{F\,[\cos\alpha - (r/R)\,]}{m\,(1+\gamma)}$, where $\cos\alpha > \dfrac{r}{R}$ (1)

(b) As static friction (fr) does not work on the spool, from the equation of the increment of mechanical energy $A_{ext} = \Delta T$.



$$A_{ext} = \frac{1}{2}mv_c^2 + \frac{1}{2}\gamma\,mR^2\,\frac{v_c^2}{R^2} = \frac{1}{2}m\,(1+\gamma)\,v_c^2$$

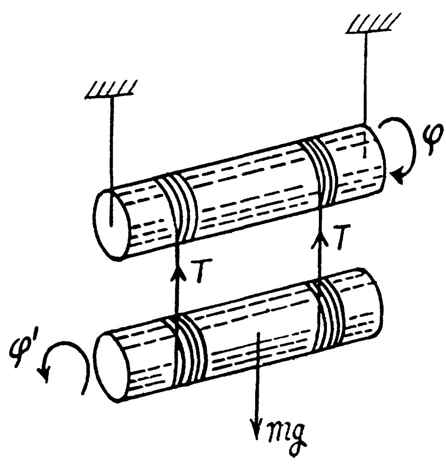
$$= \frac{1}{2}m\,(1+\gamma)\,2w_c x = \frac{1}{2}m\,(1+\gamma)\,2\,w_c\left(\frac{1}{2}w_c t^2\right)$$

$$= \frac{F^2\left(\cos\alpha - \dfrac{r}{R}\right)^2 t^2}{2\;m\,(1+\gamma)}$$

Note that at $\cos\alpha = r/R$, *there is no rolling and for* $\cos\alpha < r/R$, $w_{cx} < 0$, *i.e. the spool will move towards negative x-axis and rotate in anticlockwise sense.*

1.258 For the cylinder from the equation $N_z = I\,\beta_z$ about its stationary axis of rotation.

$$2\,Tr = \frac{mr^2}{2}\beta \quad \text{or} \quad \beta = \frac{4T}{mr} \qquad (1)$$



For the rotation of the lower cylinder from the equation $N_{cz} = I_c\,\beta_z$

$$2Tr = \frac{mr^2}{2}\beta' \quad \text{or,} \quad \beta' = \frac{4T}{mr} = \beta$$

Now for the translational motion of lower cylinder from the Eq. $F_x = mw_{cx}$:

$$mg - 2\,T = mw_c \qquad (2)$$

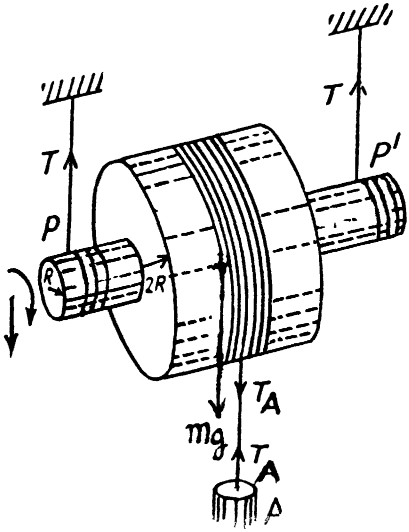
As there is no slipping of threads on the cylinders :

$$w_c = \beta' r + \beta r = 2\,\beta r \qquad (3)$$

Simultaneous solution of (1), (2) and (3) yields

$$T = \frac{mg}{10}.$$

1.259 Let us depict the forces acting on the pulley and weight A, and indicate positive direction for x and φ as shown in the figure. For the cylinder from the equation $F_x = m\, w_x$ and $N_{cz} = I_c\, \beta_z$, we get



$$Mg + T_A - 2T = M w_c \qquad (1)$$

and $$2TR + T_A (2R) = I\beta = \frac{I w_c}{R} \qquad (2)$$

For the weight A from the equation

$$F_x = m w_x$$

$$mg - T_A = m w_A \qquad (3)$$

As there is no slipping of the threads on the pulleys.

$$w_A = w_c + 2\beta R = w_c + 2 w_c = 3 w_c \qquad (4)$$

Simultaneous solutions of above four equations gives :

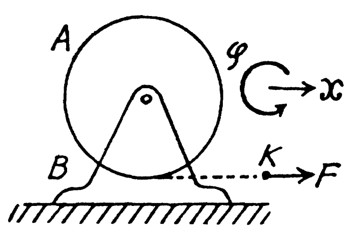
$$w_A = \frac{3(M + 3m)g}{\left(M + 9m + \frac{I}{R^2}\right)}$$

1.260 (a) For the translational motion of the system $(m_1 + m_2)$, from the equation : $F_x = m w_{cx}$

$$F = (m_1 + m_2) w_c \quad \text{or,} \quad w_c = F/(m_1 + m_2) \qquad (1)$$

Now for the rotational motion of cylinder from the equation : $N_{cx} = I_c\, \beta_z$

$$Fr = \frac{m_1 r^2}{2}\beta \quad \text{or} \quad \beta r = \frac{2F}{m_1} \qquad (2)$$



But $\quad w_K = w_c + \beta r$, So

$$w_K = \frac{F}{m_1 + m_2} + \frac{2F}{m_1} = \frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)} \qquad (3)$$

(b) From the equation of increment of mechanical energy : $\Delta T = A_{ext}$

Here $\quad \Delta T = T(t)$, so, $T(t) = A_{ext}$

As force F is constant and is directed along x-axis the sought work done.

$$A_{ext} = Fx$$

(where x is the displacement of the point of application of the force F during time interval t)

$$= F\left(\frac{1}{2} w_K t^2\right) = \frac{F^2 t^2 (3 m_1 + 2 m_2)}{2 m_1 (m_1 + m_2)} = T(t)$$

(using Eq. (3)

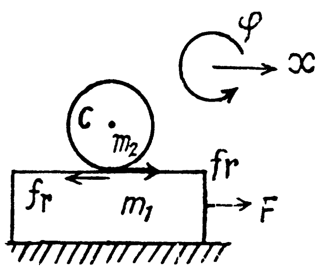
Alternate : $T(t) = T_{translation}(t) + T_{rotation}(t)$

$$= \frac{1}{2}(m_1 + m_2)\left(\frac{Ft}{(m_1 + m_2)}\right)^2 + \frac{1}{2}\frac{m_1 r^2}{2}\left(\frac{2Ft}{m_1 r}\right)^2 = \frac{F^2 t^2 (3 m_1 + 2 m_2)}{2 m_1 (m_{1+} m_2)}$$

1.261 Choosing the positive direction for x and φ as shown in Fig, let us we write the equation of motion for the sphere $F_x = m w_{cx}$ and $N_{cz} = I_c \beta_z$

$$fr = m_2 w_2 \,;\; fr\, r = \frac{2}{5} m_2 r^2 \beta$$

(w_2 is the acceleration of the C.M. of sphere.)



For the plank from the Eq. $F_x = m w_x$

$$F - f_r = m_1 w_1$$

In addition, the condition for the absence of slipping of the sphere yields the kinematical relation between the accelerations :

$$w_1 = w_2 + \beta r$$

Simultaneous solution of the four equations yields :

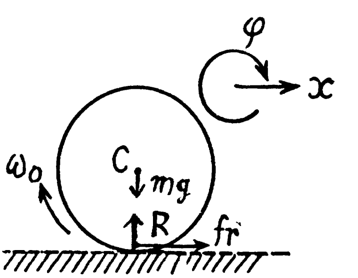
$$w_1 = \frac{F}{\left(m_1 + \frac{2}{7} m_2\right)} \quad \text{and} \quad w_2 = \frac{2}{7} w_1$$

1.262 (a) Let us depict the forces acting on the cylinder and their point of applications for the cylinder and indicate positive direction of x and φ as shown in the figure. From the equations for the plane motion of a solid $F_x = m w_{cx}$ and $N_{cz} = I_c \beta_z$:

$$k mg = m w_{cx} \quad \text{or} \quad w_{cx} = kg \tag{1}$$

$$-kmg R = \frac{mR^2}{2} \beta^z \quad \text{or} \quad \beta_z = -2\frac{kg}{R} \tag{2}$$

Let the cylinder starts pure rolling at $t = t_0$ after releasing on the horizontal floor at $t = 0$.



From the angular kinematical equation

$$\omega_z = \omega_{oz} + \beta_z t,$$

or $$\omega = \omega_0 - 2\frac{kg}{R} t \tag{3}$$

From the equation of the linear kinematics,

$$v_{cx} = v_{ocx} + w_{cx} t$$

or $$v_c = 0 + kg\, t_0 \tag{4}$$

But at the moment $t = t_0$, when pure rolling starts $v_c = \omega R$

so, $$kg\, t_0 = \left(\omega_0 - 2\frac{kg}{R} t_0\right) R$$

Thus $$t_0 = \frac{\omega_0 R}{3\, kg}$$

(b) As the cylinder pick, up speed till it starts rolling, the point of contact has a purely translatory movement equal to $\frac{1}{2} w_c\ t_0^2$ in the forward directions but there is also a backward movement of the point of contact of magnitude $(\omega_0 \tau_0 - \frac{1}{2}\beta\, t_0^2) R$. Because of slipping the net displacement is backwards. The total work done is then,

$$A_{fr} = kmg\left[\frac{1}{2} w_c\ t_0^2 - (\omega_0 t_0 + \frac{1}{2}\beta t_0^2) R\right]$$

$$= kmg\left[\frac{1}{2} kg\, t_0^2 - \frac{1}{2}\left(-\frac{2kg}{R}\right) t_0^2\, R - \omega_0 t_0 R\right]$$

$$= kmg \frac{\omega_0 R}{3kg}\left[\frac{\omega_0 R}{6} + \frac{\omega_0 R}{3} - \omega_0 R\right] = -\frac{m\omega_0^2 R^2}{6}$$

The same result can also be obtained by the work-energy theorem, $A_{fr} = \Delta T$.

1.263 Let us write the equation of motion for the centre of the sphere at the moment of breaking-off:

$$mv^2/(R+r) = mg\cos\theta,$$

where v is the velocity of the centre of the sphere at that moment, and θ is the corresponding angle (Fig.). The velocity v can be found from the energy conservation law :

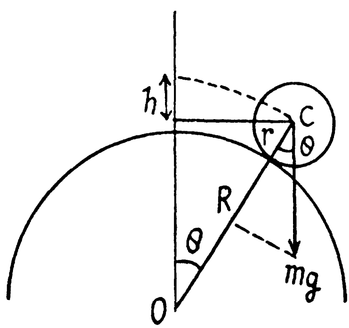
$$mgh = \frac{1}{2}\, mv^2 + \frac{1}{2}\, I\omega^2,$$

where I is the moment of inertia of the sphere relative to the axis passing through the sphere's centre. i.e. $I = \frac{2}{5} mr^2$. In addition,

$$v = \omega r\ ;\ h = (R+r)(1-\cos\theta).$$

From these four equations we obtain

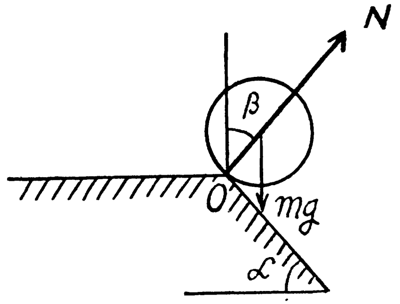
$$\omega = \sqrt{10\, g\,(R+r)\, 17\, r^2}.$$



1.264 Since the cylinder moves without sliding, the centre of the cylinder rotates about the point O, while passing through the common edge of the planes. In other words, the point O becomes the foot of the instantaneous axis of rotation of the cylinder.

It at any instant during this motion the velocity of the C.M. is v_1 when the angle (shown in the figure) is β, we have

$$\frac{m\, v_1^2}{R} = mg\cos\beta - N,$$



where N is the normal reaction of the edge

or, $\quad v_1^2 = gR\cos\beta - \dfrac{NR}{m}$ (1)

From the energy conservation law,

$$\frac{1}{2}I_0\frac{v_1^2}{R^2} - \frac{1}{2}I_0\frac{v_0^2}{R^2} = mgR\,(1-\cos\beta)$$

But $\quad I_0 = \dfrac{mR^2}{2} + mR^2 = \dfrac{3}{2}mR^2,$

(from the parallel axis theorem)

Thus, $$v_1^2 = v_0^2 + \frac{4}{3}gR\,(1-\cos\beta) \qquad (2)$$

From (1) and (2)

$$v_0^2 = \frac{gR}{3}(7\cos\beta - 4) - \frac{NR}{m}$$

The angle β in this equation is clearly smaller than or equal to α so putting $\beta = \alpha$ we get

$$v_0^2 = \frac{gR}{3}(7\cos\alpha - 4) - \frac{N_0R}{M}$$

where N_0 is the corresponding reaction. Note that $N \geq N_0$. No jumping occurs during this turning if $N_0 > 0$. Hence, v_0 must be less than

$$v_{max} = \sqrt{\frac{gR}{3}(7\cos\alpha - 4)}$$

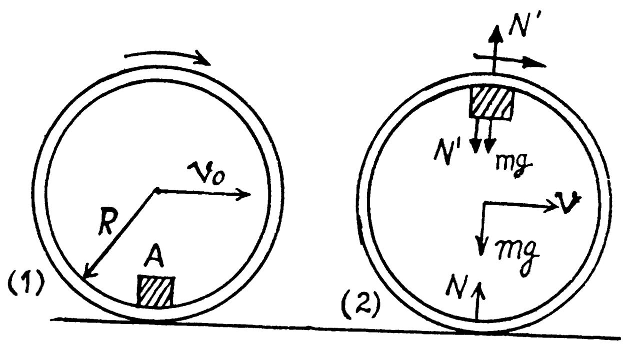
1.265 Clearly the tendency of bouncing of the hoop will be maximum when the small body A, will be at the highest point of the hoop during its rolling motion. Let the velocity of C.M. of the hoop equal v at this position. The static friction does no work on the hoop, so from conservation of mechanical energy; $E_1 = E_2$

or, $$0 + \frac{1}{2}mv_0^2 + \frac{1}{2}mR^2\left(\frac{v_0}{R}\right)^2 - mgR = \frac{1}{2}m(2v)^2 + \frac{1}{2}mv^2 + \frac{1}{2}mR^2\left(\frac{v}{R}\right)^2 + mgR$$

or, $$3v^2 = v_0^2 - 2gR \qquad (1)$$

From the equation $F_n = mw_n$ for body A at final position 2 :

$$mg + N' = m\omega^2R = m\left(\frac{v}{R}\right)^2R \qquad (2)$$



As the hoop has no acceleration in vertical direction, so for the hoop,

$$N + N' = mg \tag{3}$$

From Eqs. (2) and (3),

$$N = 2\,mg - \frac{m\,v^2}{R} \tag{4}$$

As the hoop does not bounce, $N \geq 0$ (5)

So from Eqs. (1), (4) and (5),

$$\frac{8\,g\,R - v_0^2}{3\,R} \geq 0 \quad \text{or} \quad 8\,g\,R \geq v_0^2$$

Hence $$v_0 \leq \sqrt{8\,g\,R}$$

1.266 Since the lower part of the belt is in contact with the rigid floor, velocity of this part becomes zero. The crawler moves with velocity v, hence the velocity of upper part of the belt becomes $2v$ by the rolling condition and kinetic energy of upper part $= \frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = mv^2$, which is also the sought kinetic energy, assuming that the length of the belt is much larger than the radius of the wheels.

1.267 The sphere has two types of motion, one is the rotation about its own axis and the other is motion in a circle of radius R. Hence the sought kinetic energy

$$T = \frac{1}{2} I_1\,\omega_1^2 + \frac{1}{2} I_2\,\omega_2^2 \tag{1}$$

where I_1 is the moment of inertia about its own axis, and I_2 is the moment of inertia about the vertical axis, passing through O,

But, $I_1 = \frac{2}{5} mr^2$ and $I_2 = \frac{2}{5} mr^2 + mR^2$ (using parallel axis theorem,) (2)

In addition to

$$\omega_1 = \frac{v}{r} \text{ and } \omega_2 = \frac{v}{R} \tag{3}$$

Using (2) and (3) in (1), we get $T' = \frac{7}{10}\,mv^2\left(1 + \frac{2r^2}{7R^2}\right)$

1.268 For a point mass of mass dm, looked at from C rotating frame, the equation is

$$dm\,\vec{w}' = \vec{f} + dm\,\omega^2\,\vec{r}' + 2\,dm\,(\vec{v}' \times \vec{\omega})$$

where $\vec{r}'$ = radius vector in the rotating frame with respect to rotation axis and $\vec{v}'$ = velocity in the same frame. The total centrifugal force is clearly

$$\vec{F}_{cf} = \sum dm\,\omega^2\,\vec{r}' = m\omega^2\,\vec{R}_c$$

$\vec{R}_c$ is the radius vector of the C.M. of the body with respect to rotation axis, also

$$\vec{F}_{cor} = 2m\,\vec{v}_c' \times \vec{\omega}$$

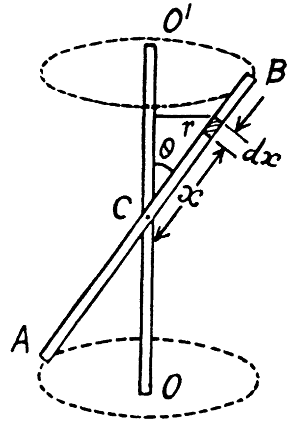
where we have used the definitions

$$m\,\vec{R}_c = \sum dm\,\vec{r}' \quad \text{and} \quad m\,\vec{v}_c' = \sum dm\,\vec{v}'$$

1.269 Consider a small element of length dx at a distance x from the point C, which is rotating in a circle of radius $r = x \sin \theta$

Now, mass of the element $= \left(\frac{m}{l}\right) dx$

So, centrifugal force acting on this element $= \left(\frac{m}{l}\right) dx\, \omega^2 x \sin \theta$ and moment of this force about C,



$$|dN| = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin \theta \cdot x \cos \theta$$

$$= \frac{m\,\omega^2}{2l} \sin 2\,\theta\, x^2\, dx$$

and hence, total moment

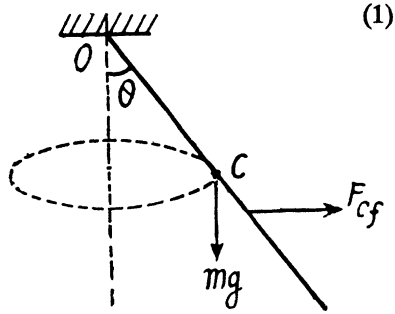
$$N = 2\int_0^{l/2} \frac{m\,\omega^2}{2l} \sin 2\,\theta\; x^2\, dx = \frac{1}{24} m\,\omega^2\, l^2 \sin 2\,\theta,$$

1.270 Let us consider the system in a frame rotating with the rod. In this frame, the rod is at rest and experiences not only the gravitational force $m\vec{g}$ and the reaction force $\vec{R}$, but also the centrifugal force $\vec{F}_{cf}$.

In the considered frame, from the condition of equilibrium i.e. $N_{0z} = 0$

or, $$N_{cf} = mg\,\frac{l}{2} \sin \theta \qquad (1)$$

where N_{cf} is the moment of centrifugal force about O. To calculate N_{cf}, let us consider an element of length dx, situated at a distance x from the point O. This element is subjected to a horizontal pseudo force $\left(\frac{m}{l}\right) dx\, \omega^2 x \sin \theta$.



The moment of this pseudo force about the axis of rotation through the point O is

$$dN_{cf} = \left(\frac{m}{l}\right) dx\, \omega^2 x \sin \theta\, x \cos \theta$$

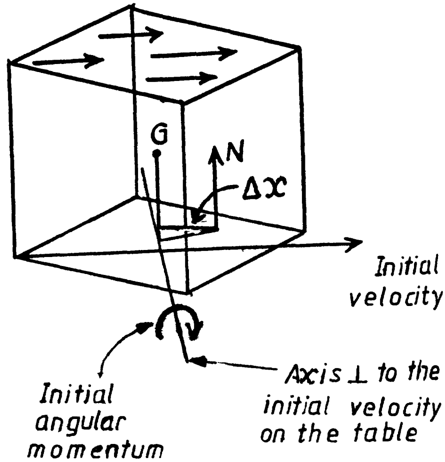
$$= \frac{m\,\omega^2}{l} \sin \theta \cos \theta\; x^2\, dx$$

So $$N_{cf} = \int_0^l \frac{m\,\omega^2}{l} \sin \theta \cos \theta\, x^2\, dx = \frac{m\,\omega^2\, l^2}{3} \sin \theta \cos \theta \qquad (2)$$

It follows from Eqs. (1) and (2) that,

$$\cos \theta = \left(\frac{3\,g}{2\,\omega^2\, l}\right) \text{ or } \theta = \cos^{-1}\left(\frac{3\,g}{2\,\omega^2\, l}\right) \qquad (3)$$

1.271 When the cube is given an initial velocity on the table in some direction (as shown) it acquires an angular momentum about an axis on the table perpendicular to the initial velocity and (say) just below the C.G.. This angular momentum will disappear when the cube stops and this can only by due to a torque. Frictional forces cannot do this by themselves because they act in the plain containing the axis. But if the force of normal reaction act eccentrically (as shown), their torque can bring about the vanishing of the angular momentum. We can calculate the distance Δx between the point of application of the normal reaction and the C.G. of the cube as follows. Take the moment about C.G. of all the forces. This must vanish because the cube does not turn or tumble on the table. Then if the force of friction is fr



$$fr\,\frac{a}{2} = N\,\Delta x$$

But $N = mg$ and $fr = kmg$, so

$$\Delta x = ka/2$$

1.272 In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence, it follows that

$$\frac{1}{2}\frac{Ml^2}{3}\omega_0^2 = \frac{1}{2}m(\omega^2 l^2 + v'^2) + \frac{1}{2}\frac{Ml^2}{3}\omega^2$$

(ω is the final angular velocity of the rod)

and
$$\frac{Ml^2}{3}\omega_0 = \frac{Ml^2}{3}\omega + ml^2\omega$$

From these equations we obtain

$$\omega = \omega_0 \Big/ \left(1 + \frac{3M}{M}\right) \text{ and}$$

$$v' = \omega_0 l \Big/ \sqrt{1 + 3m/M}$$

1.273 Due to hitting of the ball, the angular impulse received by the rod about the C.M. is equal to $p\,\frac{1}{2}$. If ω is the angular velocity acquired by the rod, we have

$$\frac{ml^2}{12}\omega = \frac{pl}{2} \text{ or } \omega = \frac{6p}{ml} \tag{1}$$

In the frame of C.M., the rod is rotating about an axis passing through its mid point with the angular velocity ω. Hence the force exerted by one half on the other = mass of one half × acceleration of C.M. of that part, in the frame of C.M.

$$= \frac{m}{2}\left(\omega^2\frac{l}{4}\right) = m\frac{\omega^2 l}{8} = \frac{9p^2}{2ml} = 9\text{ N}$$

1.274 (a) In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence it follows that

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^2$$

and $$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{Ml^2}{3}\omega$$

From these equations we obtain

$$v' = \left(\frac{3m-4M}{3m+4M}\right)v. \quad \text{and} \quad \omega = \frac{4v}{l(1+4m/3M)}$$

As $\vec{v}' \uparrow\uparrow \vec{v}$, so in vector form $\vec{v}' = \left(\frac{3m-4M}{3m+4M}\right)\vec{v}$

(b) Obviously the sought force provides the centripetal acceleration to the C.M. of the rod and is

$$F_n = mw_{cn}$$

$$= M\omega^2\frac{l}{2} = \frac{8Mv^2}{l(1+4M/3m)^2}$$

1.275 (a) About the axis of rotation of the rod, the angular momentum of the system is conserved. Thus if the velocity of the flying bullet is v.

$$mvl = \left(ml^2 + \frac{Ml^2}{3}\right)\omega$$

$$\omega = \frac{mv}{\left(m+\frac{M}{3}\right)l} \approx \frac{3mv}{Ml} \quad \text{as} \quad m \ll M \tag{1}$$

Now from the conservation of mechanical energy of the system (rod with bullet) in the uniform field of gravity

$$\frac{1}{2}\left(ml^2 + \frac{Ml^2}{3}\right)\omega^2 = (M+m)g\frac{l}{2}(1-\cos\alpha) \tag{2}$$

[because C.M. of rod raises by the height $\frac{l}{2}(1-\cos\alpha)$]

Solving (1) and (2), we get

$$v = \left(\frac{M}{m}\right)\sqrt{\frac{2}{3}gl}\,\sin\frac{\alpha}{2} \quad \text{and} \quad \omega = \sqrt{\frac{6g}{l}}\,\sin\frac{\alpha}{2}$$

(b) Sought $\Delta p = \left[m(\omega l) + M\left(\omega\frac{l}{2}\right)\right] - mv$

where ωl is the veloccity of the bullet and $\omega\frac{l}{2}$ equals the velocity of C.M. of the rod after the impact. Putting the value of v and ω we get

$$\Delta p \approx \frac{1}{2}mv = M\sqrt{\frac{gl}{6}}\,\sin\frac{\alpha}{2}$$

This is caused by the reaction at the hinge on the upper end.

(c) Let the rod starts swinging with angular velocity ω', in this case. Then, like part (a)

$$mvx = \left(\frac{Ml^2}{3} + mx^2\right)\omega' \quad \text{or} \quad \omega' \approx \frac{3mvx}{Ml^2}$$

Final momentum is

$$p_f = m\,x\,\omega' + \int_0^l y\,\omega' \frac{M}{l}\,dy \approx \frac{M}{2}\,\omega'\,l \approx \frac{3}{2}\,m\,v\,\frac{x}{l}$$

So, $$\Delta p \approx p_f - p_i \approx m\,v\left(\frac{3\,x}{2l} - 1\right)$$

This vanishes for $$x \approx \frac{2}{3}\,l$$

1.276 (a) As force F on the body is radial so its angular momentum about the axis becomes zero and the angular momentum of the system about the given axis is conserved. Thus

$$\frac{MR^2}{2}\,\omega_0 + m\,\omega_0 R^2 = \frac{MR^2}{2}\,\omega \quad \text{or} \quad \omega = \omega_0\left(1 + \frac{2m}{M}\right)$$

(b) From the equation of the increment of the mechanical energy of the system :

$$\Delta T = A_{ext}$$

$$\frac{1}{2}\frac{MR^2}{2}\,\omega^2 - \frac{1}{2}\left(\frac{MR^2}{2} + m\,R^2\right)\omega_0^2 = A_{ext}$$

Putting the value of ω from part (a) and solving we get

$$A_{ext} = \frac{m\,\omega_0^2\,R^2}{2}\left(1 + \frac{2m}{M}\right)$$

1.277 (a) Let z be the rotation axis of disc and φ be its rotation angle in accordance with right-hand screw rule (Fig.). (φ and φ' are to be measured in the same sense algebraically.) As M_z of the system (disc + man) is conserved and $M_{z(\,initial\,)} = 0$, we have at any instant,

$$0 = \frac{m_2 R^2}{2}\frac{d\,\varphi}{dt} + m_1\left[\left(\frac{d\,\varphi'}{dt}\right)R + \left(\frac{d\,\varphi}{dt}\right)R\right]R$$

or, $$d\,\varphi = \left[-\frac{m_1}{m_1 + (\,m_2/2\,)}\right]d\,\varphi'$$

On integrating $$\int_0^{\varphi} d\,\varphi = -\int_0^{\varphi'}\left(\frac{m_1}{m_1 + (\,m_2/2\,)}\right)d\,\varphi'$$

or, $$\varphi = -\left(\frac{m_1}{m_1 + \dfrac{m_2}{2}}\right)\varphi' \tag{1}$$

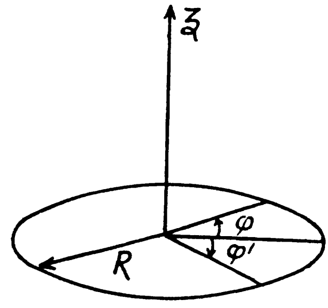
This gives the total angle of rotation of the disc.

(b) From Eq. (1)

$$\frac{d\varphi}{dt} = -\left(\frac{m_1}{m_1+\frac{m_2}{2}}\right)\frac{d\varphi'}{dt} = -\left(\frac{m_1}{m_1+\frac{m_2}{2}}\right)\frac{v'(t)}{R}$$

Differentiating with respect to time

$$\frac{d^2\varphi}{dt^2} = -\left(\frac{m_1}{m_1+\frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$



Thus the sought force moment from the Eq. $N_z = I\beta_z$

$$N_z = \frac{m_2R^2}{2}\frac{d^2\varphi}{dt^2} = -\frac{m_2R^2}{2}\left(\frac{m_1}{m_1+\frac{m_2}{2}}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$

Hence $$N_z = -\frac{m_1m_2R}{2m_1+m_2}\frac{dv'(t)}{dt}$$

1.278 (a) Frome the law of conservation of angular momentum of the system relative to vertical axis z, it follows that:

$$I_1\omega_{1z} + I_2\omega_{2z} = (I_1+I_2)\,\omega_z$$

Hence $$\omega_z = (I_1\omega_{1z} + I_2\omega_{2z})\,/\,(I_1+I_2) \qquad (1)$$

Not that for $\omega_z > 0$, the corresponding vector $\vec{\omega}$ coincides with the poitive direction to the z axis, and vice versa. As both discs rotates about the same vertical axis z, thus in vector form.

$$\vec{\omega} = I_1\vec{\omega}_1 + I_2\vec{\omega}_2\,/\,(I_1+I_2)$$

However, the problem makes sense only if $\vec{\omega}_1 \uparrow\uparrow \vec{\omega}_2$ or $\vec{\omega}_1 \uparrow\downarrow \vec{\omega}_2$

(b) From the equation of increment of mechanical energy of a system: $A_{fr} = \Delta T$.

$$= \frac{1}{2}(I_1+I_2)\,\omega_z^2 - \frac{1}{2}I_1\omega_{1z}^2 + \frac{1}{2}I_2\omega_{2z}^2$$

Using Eq. (1)

$$A_{fr} = -\frac{I_1I_2}{2(I_1+I_2)}(\omega_{1z}-\omega_{2z})^2$$

1.279 For the closed system (disc + rod), the angular momentum is conserved about any axis. Thus from the conservation of angular momentum of the system about the rotation axis of rod passing through its C.M. gives :

$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{\eta ml^2}{12}\omega \qquad (1)$$

(v' is the final velocity of the disc and ω angular velocity of the rod)

For the closed system linear momentum is also conserved. Hence

$$mv = mv' + \eta\, mv_c \qquad (2)$$

(where v_c is the velocity of C.M. of the rod)

From Eqs (1) and (2) we get

$$v_c = \frac{l\,\omega}{3} \quad \text{and} \qquad v - v' = \eta v_c$$

Applying conservation of kinetic energy, as the collision is elastic

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^{\,2} + \frac{1}{2}\eta m v_c^{\,2} + \frac{1}{2}\,\frac{\eta m l^2}{12}\,\omega^2 \qquad (3)$$

or $\qquad v^2 - v'^2 = 4\eta v_c^{\,2}$ and hence $v + v' = 4v_c$

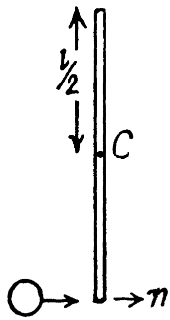
Then

$$v' = \frac{4-\eta}{4+\eta}\, v \text{ and } \omega = \frac{12\, v}{(4+\eta)\, l}$$

Vectorially, noting that we have taken $\vec{v}'$ parallel to $\vec{v}$

$$\vec{u}' = \left(\frac{4-\eta}{4+\eta}\right)\vec{v}$$

So, $\vec{u}' = 0$ for $\eta = 4$ and $\vec{u}' \downarrow\uparrow \vec{v}$ for $\eta > 4$



1.280 See the diagram in the book (Fig. 1.72)

(a) When the shaft BB' is turned through 90° the platform must start turning with angular velocity Ω so that the angular momentum remains constant. Here

$$(I + I_0)\,\Omega = I_0\,\omega_0 \quad \text{or,} \quad \Omega = \frac{I_0\,\omega_0}{I + I_0}$$

The work performed by the motor is therefore

$$\frac{1}{2}(I + I_0)\,\Omega^2 = \frac{1}{2}\,\frac{I_0^2\,\omega_0^2}{I + I_0}$$

If the shaft is turned through 180°, angular velocity of the sphere changes sign. Thus from conservation of angular momentum,

$$I\,\Omega - I_0\,\omega_0 = I_0\,\omega_0$$

(Here $-I_0\,\omega_0$ is the complete angular momentum of the sphere i. e. we assume that the angular velocity of the sphere is just $-\omega_0$). Then

$$\Omega = 2I_0\,\frac{\omega_0}{I}$$

and the work done must be,

$$\frac{1}{2}I\,\Omega^2 + \frac{1}{2}I_0\,\omega_0^2 - \frac{1}{2}I_0\,\omega_0^2 = \frac{2I_0^2\,\omega_0^2}{I}$$

(b) In the case (a), first part, the angular momentum vector of the sphere is precessing with angular velocity Ω. Thus a torque,

$$I_0 \omega_0 \Omega = \frac{I_0^2 \omega_0^2}{I + I_0} \text{ is needed.}$$

1.281 The total centrifugal force can be calculated by,

$$\int_0^{l_0} \frac{m}{l_0} \omega^2 x \, dx = \frac{1}{2} m \, l_0 \, \omega^2$$

Then for equilibrium,

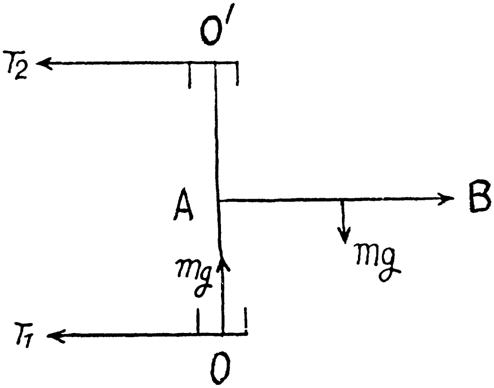
$$(T_2 - T_1) \frac{l}{2} = mg \frac{l_0}{2}$$

and, $T_2 + T_1 = \frac{1}{2} m \, l_0 \, \omega^2$

Thus T_1 vanishes, when

$$\omega^2 = \frac{2g}{l}, \quad \omega = \sqrt{\frac{2g}{l}} = 6 \text{ rad/s}$$

Then $T_2 = mg \frac{l_0}{l} = 25 \text{ N}$



1.282 See the diagram in the book (Fig. 1.71).

(a) The angular velocity $\vec{\omega}$ about OO' can be resolved into a component parallel to the rod and a component $\omega \sin\theta$ perpendicular to the rod through C. The component parallel to the rod does not contribute so the angular momentum

$$M = I \omega \sin\theta = \frac{1}{12} m \, l^2 \omega \sin\theta$$

Also, $$M_z = M \sin\theta = \frac{1}{12} m \, l^2 \omega \sin^2 \theta$$

This can be obtained directly also,

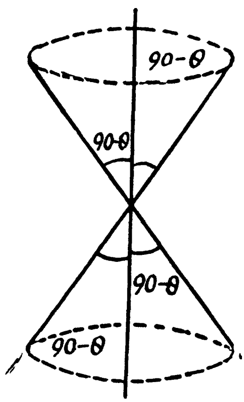
(b) The modulus of $\vec{M}$ does not change but the modulus of the change of $\vec{M}$ is $|\Delta \vec{M}|$.

$$|\Delta \vec{M}| = 2M \sin(90 - \theta) = \frac{1}{12} m \, l^2 \omega \sin 2\theta$$

(c) Here $M_{\perp} = M \cos\theta = I \omega \sin\theta \cos\theta$

Now $$\left| \frac{d\vec{M}}{dt} \right| = I \omega \sin\theta \cos\theta \frac{\omega \, dt}{dt} = \frac{1}{24} m l^2 \omega^2 \sin^2\theta$$

as $\vec{M}$ precesses with angular velocity ω.



1.283 Here $M = I\omega$ is along the symmetry axis. It has two components, the part $I\omega\cos\theta$ is constant and the part $M_\perp = I\omega\sin\theta$ presesses, then

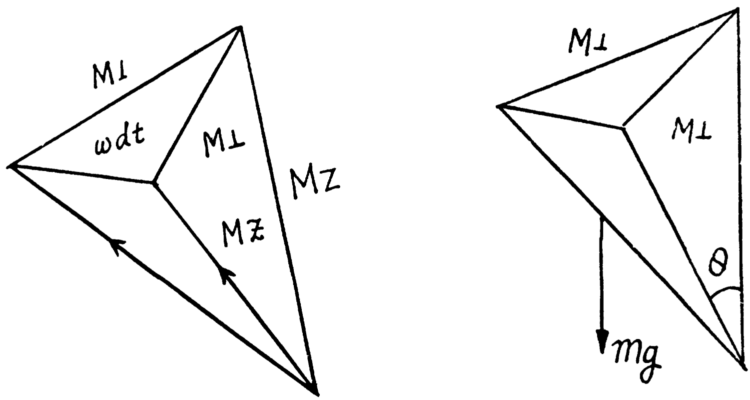
$$\left|\frac{d\vec{M}}{dt}\right| = I\omega\sin\theta\,\omega' = mgl\sin\theta$$

or, $$\omega' = \text{precession frequency} = \frac{mgl}{I\omega} = 0{\cdot}7 \text{ rad/s}$$

(b) This force is the centripetal force due to precession. It acts inward and has the magnitude

$$|\vec{F}| = \left|\sum m_i\omega'^2\vec{\rho_i}\right| = m\omega'^2 l\sin\theta = 12 \text{ mN}.$$

$\vec{\rho_i}$ is the distance of the $i\,th$ element from the axis. This is the force that the table will exert on the top. See the diagram in the answer sheet



1.284 See the diagram in the book (Fig. 1.73).

The moment of inertia of the disc about its symmentry axis is $\frac{1}{2}mR^2$. If the angular velocity of the disc is ω then the angular momentum is $\frac{1}{2}mR^2\omega$. The precession frequency being $2\pi n$,

we have $$\left|\frac{d\vec{M}}{dt}\right| = \frac{1}{2}mR^2\omega \times 2\pi n$$

This must equal $m(g + w)l$, the effective gravitational torques (g being replaced by $g + w$ in the elevator). Thus,

$$\omega = \frac{(g+w)l}{\pi R^2 n} = 300 \text{ rad/s}$$

1.285 The effective g is $\sqrt{g^2 + w^2}$ inclined at angle $\tan^{-1}\frac{w}{g}$ with the vertical. Then with reference to the new " vertical" we proceed as in problem 1·283. Thus

$$\omega' = \frac{ml\sqrt{g^2 + w^2}}{I\omega} = 0{\cdot}8 \text{ rad/s.}$$

The vector $\vec{\omega}'$ forms an angle $\theta = \tan^{-1}\frac{w}{g} = 6°$ with the normal vertical.

1.286 The moment of inertia of the sphere is $\frac{2}{5}mR^2$ and hence the value of angular momentum is $\frac{2}{5}mR^2\omega$. Since it precesses at speed ω' the torque required is

$$\frac{2}{5}mR^2\omega\,\omega' = F'\,l$$

So, $$F' = \frac{2}{5}mR^2\omega\,\omega'/l = 300 \text{ N}$$

(The force F' must be vertical.)

1.287 The moment of inertia is $\frac{1}{2}mr^2$ and angular momentum is $\frac{1}{2}mr^2\omega$. The axle oscillates about a horizontal axis making an instantaneous angle.

$$\varphi = \varphi_m \sin\frac{2\pi t}{T}$$

This means that there is a variable precession with a rate of precession $\frac{d\varphi}{dt}$. The maximum value of this is $\frac{2\pi\varphi_m}{T}$. When the angle between the axle and the axis is at its maximum value, a torque $I\,\omega\,\Omega$

$$= \frac{1}{2}mr^2\omega\,\frac{2\pi\varphi_m}{T} = \frac{\pi\,mr^2\,\omega\varphi_m}{T} \text{ acts on it.}$$

The corresponding gyroscopic force will be $\frac{\pi\,mr^2\,\omega\varphi_m}{lT} = 90$ N

1.288 The revolutions per minute of the flywheel being n, the angular momentum of the flywheel is $I \times 2\pi n$. The rate of precession is $\frac{v}{R}$

Thus $N = 2\pi I N V/R = 5{\cdot}97$ kN. m.

1.289 As in the previous problem a couple $2\pi Inv/R$ must come in play. This can be done if a force, $\frac{2\pi Inv}{Rl}$ acts on the rails in opposite directions in addition to the centrifugal and other forces. The force on the outer rail is increased and that on the inner rail decreased. The additional force in this case has the magnitude 1·4 kN. m.

1.6 ELASTIC DEFORMATIONS OF A SOLID BODY

1.290 Variation of length with temperature is given by

$$l_t = l_0 (1 + \alpha \Delta t) \text{ or } \frac{\Delta l}{l_0} = \alpha \Delta t = \varepsilon \quad (1)$$

But $$\varepsilon = \frac{\sigma}{E},$$

Thus $\sigma = \alpha \Delta t E$, which is the sought stress of pressure.

Putting the value of α and E from Appendix and taking $\Delta t = 100°C$, we get

$$\sigma = 2{\cdot}2 \times 10^3 \text{ atm.}$$

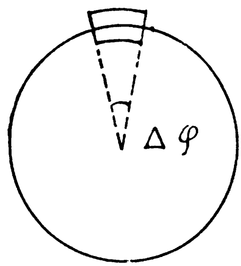
1.291 (a) Consider a transverse section of the tube and concentrate on an element which subtends an angle $\Delta\varphi$ at the centre. The forces acting on a portion of length Δl on the element are (1) tensile forces side ways of magnitude $\sigma \Delta r \Delta l$.

The resultant of these is

$$2\sigma \Delta r \Delta l \sin \frac{\Delta\varphi}{2} \approx \sigma \Delta r \Delta l \Delta\varphi$$

radially towards the cente.

(2) The force due to fluid pressure $= p r \Delta\varphi \Delta l$



Since these balance, we get $p_{max} \approx \sigma_m \frac{\Delta r}{r}$

where σ_m is the maximum tensile force.

Putting the values we get $p_{max} = 19{\cdot}7$ atmos.

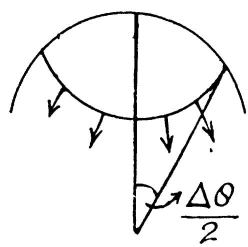
(b) Consider an element of area $dS = \pi (r \Delta\theta/2)^2$ about z-axis chosen arbitrarily. There are tangential tensile forces all around the ring of the cap. Their resultant is

$$\sigma \left[2\pi \left(r \frac{\Delta\theta}{2} \right) \Delta r \right] \sin \frac{\Delta\theta}{2}$$

Hence in the limit

$$p_m \pi \left(\frac{r\Delta\theta}{2} \right)^2 = \sigma_m \pi \left(\frac{r\Delta\theta}{2} \right) \Delta r \, \Delta\theta$$

or $$p_m = \frac{2\sigma_m \Delta r}{r} = 39{\cdot}5 \text{ atmos.}$$



1.292 Let us consider an element of rod at a distance x from its rotation axis (Fig.). From Newton's second law in projection form directed towards the rotation axis

$$-dT = (dm)\,\omega^2 x = \frac{m}{l} \omega^2 x \, dx$$

On integrating

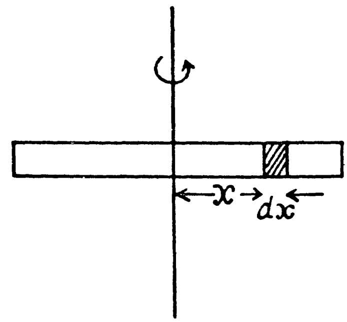
$$-T = \frac{m\omega^2}{l} \frac{x^2}{2} + C \text{ (constant)}$$

But at $x = \pm \frac{l}{2}$ or free end, $T = 0$

Thus $0 = \frac{m\omega^2}{2}\frac{l^2}{4} + C$ or $C = -\frac{m\omega^2 l}{8}$

Hence $T = \frac{m\omega^2}{2}\left(\frac{l}{4} - \frac{x^2}{l}\right)$

Thus $T_{max} = \frac{m\omega^2 l}{8}$ (at mid point)



Condition required for the problem is

$$T_{max} = S\,\sigma_m$$

So, $\frac{m\omega^2 l}{8} = S\,\sigma_m$ or $\omega = \frac{2}{l}\sqrt{\frac{2\sigma_m}{\rho}}$

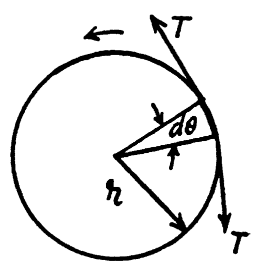
Hence the sought number of r p s

$$n = \frac{\omega}{2\pi} = \frac{1}{\pi l}\sqrt{\frac{2\sigma_m}{\rho}}$$ [using the table $n = 0{\cdot}8 \times 10^2$rps]

1.293 Let us consider an element of the ring (Fig.). From Newton's law $F_n = mw_n$ for this element, we get,

$$Td\theta = \left(\frac{m}{2\pi}d\theta\right)\omega^2 r$$ [see solution of 1.93or 1.92]

So, $T = \frac{m}{2\pi}\omega^2 r$



Condition for the problem is :

$$\frac{T}{\pi r^2} \le \sigma_m \quad \text{or,} \quad \frac{m\omega^2 r}{2\pi^2 r^2} \le \sigma_m$$

or, $\omega^2_{max} = \frac{2\pi^2 \sigma_m r}{\pi r^2 (2\pi r\rho)} = \frac{\sigma_m}{\rho r^2}$

Thus sought number of rps

$$n = \frac{\omega_{max}}{2\pi} = \frac{1}{2\pi r}\sqrt{\frac{\sigma_m}{\rho}}$$

Using the table of appendices $n = 23$rps

1.294 Let the point O desend by the distance x (Fig.). From the condition of equilibrium of point O.

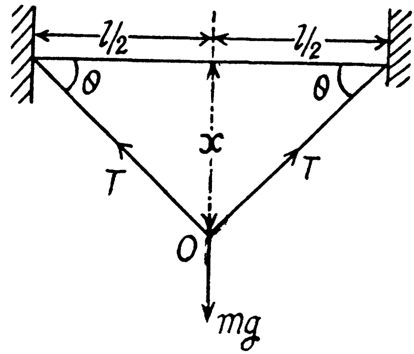
$$2T\sin\theta = mg \quad \text{or} \quad T = \frac{mg}{2\sin\theta} = \frac{mg}{2x}\sqrt{(l/2)^2 + x^2} \qquad (1)$$

Now, $\frac{T}{\pi(d/2)^2} = \sigma = \varepsilon E$ or $T = \varepsilon E \pi \frac{d^2}{4}$ (2)

(σ here is stress and ε is strain.)

In addition to it,

$$\varepsilon = \frac{\sqrt{(l/2)^2 + x^2} - \frac{l}{2}}{l/2} = \sqrt{1 + \left(\frac{2x}{l}\right)^2} - 1 \quad (3)$$



From Eqs. (1), (2) and (3)

$$x - \frac{x}{\sqrt{1 + \left(\frac{2x}{l}\right)^2}} = \frac{mgl}{\pi E d^2} \quad \text{as } x << l$$

So, $\frac{4x^3}{2l^2} \approx \frac{mgl}{\pi E d^2}$

or, $x = l\left(\frac{mg}{2\pi E d^2}\right)^{1/3} = 2{\cdot}5 \text{ cm}$

1.295 Let us consider an element of the rod at a distance x from the free end (Fig.). For the considered element '$T - T$' are internal restoring forces which produce elongation and dT provides the acceleration to the element. For the element from Newton's law :

$$dT = (dm)\, w = \left(\frac{m}{l} dx\right) \frac{F_o}{m} = \frac{F_o}{l} dx$$

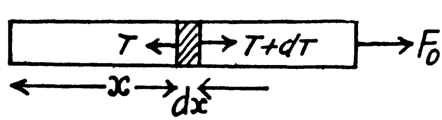
As free end has zero tension, on integrating the above expression,

$$\int_0^T dT = \frac{F_o}{l} \int_0^x dx \quad \text{or} \quad T = \frac{F_o}{l} x$$

Elongation in the considered element of lenght dx :

$$\partial \xi = \frac{\sigma}{E}(x)\; dx = \frac{T}{SE} dx = \frac{F_{xo} x dx}{SEL}$$

Thus total elengation $\xi = \frac{F_o}{SEl} \int_x^l x\, dx = \frac{F_o l}{2SE}$



Hence the sought strain

$$\sigma = \frac{\xi}{l} = \frac{F_o}{2SE}$$

1.296 Let us consider an element of the rod at a distance r from it's rotation axis. As the element rotates in a horizontal circle of radius r, we have from Newton's second law in projection form directed toward the axis of rotation :

$$T - (T + dT) = (dm)\, \omega^2 r$$

or,
$$-dT = \left(\frac{m}{l} dr\right) \omega^2 r = \frac{m}{l} \omega^2 r\, dr$$

At the free end tension becomes zero. Integrating the above experession we get, thus

$$-\int_T^0 dT = \frac{m}{l}\omega^2 \int_r^l r\,dr$$

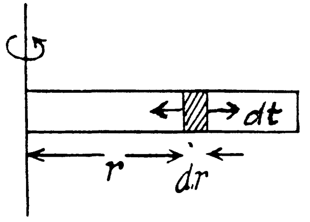
Thus $$T = \frac{m\omega^2}{l}\left(\frac{l^2 - r^2}{2}\right) = \frac{m\,\omega^2\,l}{2}\left(1 - \frac{r^2}{l^2}\right)$$

Elongation in elemental length dr is given by :

$$\partial\,\xi = \frac{\sigma\,(r)}{E}\,dr = \frac{T}{SE}\,dr$$

(where S is the cross sectional area of the rod and T is the tension in the rod at the considered element)

or, $$\partial\,\xi = \frac{m\,\omega^2\,l}{2\,SE}\left(1 - \frac{r^2}{l^2}\right)dr$$



Thus the sought elongation

$$\xi = \int d\,\xi = \frac{m\,\omega^2\,l}{2\,SE}\int_0^l \left(1 - \frac{r^2}{l^2}\right)dr$$

or, $$\xi = \frac{m\,\omega^2\,l}{2\,SE}\,\frac{2l}{3} = \frac{(Sl\,\rho)}{3\,SE}\,\omega^2\,l^3$$

$$= \frac{1}{3}\,\frac{\rho\,\omega^2\,l^3}{E}$$ (where ρ is the density of the copper.)

1.297 Volume of a solid cylinder

$$V = \pi\,r^2\,l$$

So, $$\frac{\Delta V}{V} = \frac{\pi\,2\,r\,\Delta\,r\,l}{\pi\,r^2\,l} + \frac{\pi\,r^2\,\Delta\,l}{\pi\,r^2\,l} = \frac{2\,\Delta\,r}{r} + \frac{\Delta\,l}{l} \qquad (1)$$

But longitudinal strain $\Delta l/l$ and accompanying lateral strain $\Delta\,r/r$ are related as

$$\frac{\Delta\,r}{r} = -\mu\,\frac{\Delta\,l}{l} \qquad (2)$$

Using (2) in (1), we get :

$$\frac{\Delta V}{V} = \frac{\Delta l}{l}\,(1 - 2\,\mu) \qquad (3)$$

But $$\frac{\Delta l}{l} = \frac{-F/\pi\,r^2}{E}$$

(Because the increment in the length of cylinder Δl is negative)

So, $$\frac{\Delta V}{V} = \frac{-F}{\pi\,r^2\,E}\,(1 - 2\,\mu)$$

Thus, $$\Delta V = \frac{-Fl}{E}(1-2\mu)$$

Negative sign means that the volume of the cylinder has decreased.

1.298 (a) As free end has zero tension, thus the tension in the rod at a vestical distance y from its lower end

$$T = \frac{m}{l} g y \tag{1}$$

Let ∂l be the elongation of the element of length dy, then

$$\partial l = \frac{\sigma(y)}{E} dy$$

$$= \frac{T}{SE} dy = \frac{mgydy}{SlE} = \rho\, gydy/E \quad \text{(where } \rho \text{ is the density of the copper)}$$

Thus the sought elongation

$$\Delta l = \int \partial l = \rho g \int_0^l \frac{y\,dy}{E} = \frac{1}{2} \rho g l^2/E \tag{2}$$

(b) If the longitudinal (tensile) strain is $\varepsilon = \frac{\Delta l}{l}$, the accompanying lateral (compressive) strain is given by

$$\varepsilon' = \frac{\Delta r}{r} = -\mu\varepsilon \tag{3}$$

Then since $V = \pi r^2 l$ we have

$$\frac{\Delta V}{V} = \frac{2\Delta r}{r} + \frac{\Delta l}{l}$$

$$= (1-2\mu)\frac{\Delta l}{l} \quad \text{[Using (3)]}$$

where $\frac{\Delta l}{l}$ is given in part (a), μ is the Poisson ratio for copper.

1.299 Consider a cube of unit length before pressure is applied. The pressure acts on each face. The pressures on the opposite faces constitute a tensile stress producing longitudianl compression and lateral extension. The compressions is $\frac{p}{E}$ and the lateral extension is $\mu\frac{p}{E}$

The net result is a compression

$$\frac{p}{E}(1-2\mu) \quad \text{in each side.}$$

Hence $\frac{\Delta V}{V} = -\frac{3p}{E}(1-2\mu)$ because from symmetry $\frac{\Delta V}{V} = 3\frac{\Delta l}{l}$

(b) Let us consider a cube under an equal compressive stress σ, acting on all its faces.

Then, $$\text{volume strain} = -\frac{\Delta V}{V} = \frac{\sigma}{k}, \qquad (1)$$

where k is the bulk modulus of elasticity.

So $$\frac{\sigma}{k} = \frac{3\sigma}{E}(1-2\mu)$$

or, $$E = 3k(1-2\mu) = \frac{3}{\beta}(1-2\mu)\left(\text{as } k = \frac{1}{\beta}\right)$$

$\mu \le \frac{1}{2}$ if E and β are both to remain positive.

1.300 A beam clamped at one end and supporting an applied load at the free end is called a cantilever. The theory of cantilevers is discussed in advanced text book on mechanics. The key result is that elastic forces in the beam generate a couple, whose moment, called the moment of resistances, balances the external bending moment due to weight of the beam, load etc. The moment of resistance, also called internal bending moment (I.B.M) is given by

$$\text{I.B.M.} = EI/R$$

Here R is the radius of curvature of the beam at the representative point (x, y). I is called the geometrical moment of inertia

$$I = \int z^2\, dS$$

of the cross section relative to the axis passing through the netural layer which remains unstretched. (Fig.1.). The section of the beam beyond P exerts the bending moment $N(x)$ and we have,

$$\frac{EI}{R} = N(x)$$

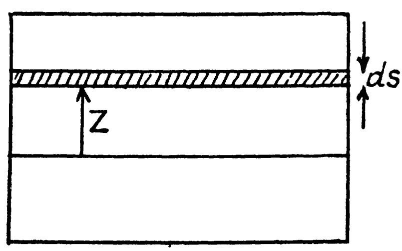
If there is no load other than that due to the weight of the beam, then

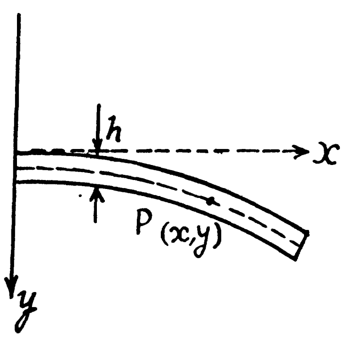
$$N(x) = \frac{1}{2}\rho g (l-x)^2 bh$$

where ρ = density of steel.

Hence, at $x = 0$

$$\left(\frac{I}{R}\right)_0 = \frac{\rho g l^2 b h}{2EI}$$





Here b = width of the beam perpendicular to paper.

Also, $$I = \int_{-h/2}^{h/2} z^2\, b\,dz = \frac{bh^3}{12}.$$

Hence, $$\left(\frac{1}{R}\right)_0 = \frac{6\rho g l^2}{Eh^2} = (0.121 \text{ km})^{-1}$$

1.301 We use the equation given above and use the result that when y is small

$$\frac{1}{R} \approx \frac{d^2 y}{dx^2}. \text{ Thus, } \frac{d^2 y}{dx^2} = \frac{N(x)}{EI}$$

(a) Here $N(x) = N_0$ is a constant. Then integration gives,

$$\frac{dy}{dx} = \frac{N_0 x}{EI} + C_1$$

But $\left(\frac{dy}{dx}\right) = 0$ for $x = 0$, so $C_1 = 0$. Integrating again,

$$y = \frac{N_0 x^2}{2EI}$$

where we have used $y = 0$ for $x = 0$ to set the constant of integration at zero. This is the equation of a parabola. The sag of the free end is

$$\lambda = y(x = l) = \frac{N_0 l^2}{2EI}$$

(b) In this case $N(x) = F(l - x)$ because the load F at the extremity is balanced by a similar force at F directed upward and they constitute a couple. Then

$$\frac{d^2 y}{dx^2} = \frac{F(l-x)}{EI}.$$

Integrating, $$\frac{dy}{dx} = \frac{F(lx - x^2/2)}{EI} + C_1$$

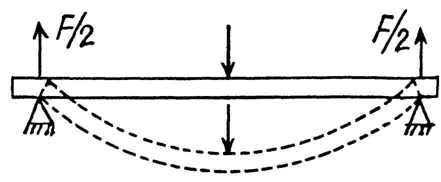
As before $C_1 = 0$. Integrating again, using $y = 0$ for $x = 0$

$$y = \frac{F\left(\frac{lx^2}{2} - \frac{x^3}{6}\right)}{EI} \quad \text{here } \lambda = \frac{Fl^3}{3EI}$$

Here for a square cross section

$$I = \int_{-a/2}^{a/2} z^2 a\, dz = a^4/12.$$

1.302 One can think of it as analogous to the previous case but with a beam of length $l/2$ loaded upward by a force $F/2$.



Thus $$\lambda = \frac{F l^3}{48 EI},$$

On using the last result of the previous problem.

1.303 (a) In this case $N(x) = \frac{1}{2} \rho g b h (l - x)^2$ where $b =$ width of the girder.

Also $$I = b h^3/12. \text{ Then,}$$

$$\frac{E\,b\,h^2}{12}\frac{d^2y}{dx^2} = \frac{\rho\,g\,b\,h}{2}(l^2 - 2l\,x + x^2).$$

Integrating, $$\frac{dy}{dx} = \frac{6\,\rho g}{E\,h^2}\left(l^2 x - lx^2 + \frac{x^3}{3}\right)$$

using $\frac{dy}{dx} = 0$ for $x = 0$. Again integrating

$$y = \frac{6\,\rho g}{Eh^2}\left(\frac{l^2 x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right)$$

Thus $$\lambda = \frac{6\,\rho\,g\,l^4}{Eh^2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{12}\right)$$

$$= \frac{6\,\rho\,g\,l^4}{Eh^2}\,\frac{3}{12} = \frac{3\,\rho\,g\,l^4}{2\,Eh^2}$$

(b) As before, $EI\frac{d^2y}{dx^2} = N(x)$ where $N(x)$ is the bending moment due to section PB.

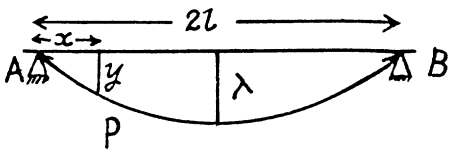
This bending moment is clearly

$$N = \int_x^{2l} w\,d\xi\,(\xi - x) - wl\,(2l - x)$$

$$= w\left(2l^2 - 2xl + \frac{x^2}{2}\right) - wl\,(2l - x) = w\left(\frac{x^2}{2} - xl\right)$$

(Here $w = \rho\,g\,b\,h$ is weight of the beam per unit length)

Now integrating, $EI\frac{dy}{dx} = w\left(\frac{x^3}{6} - \frac{x^2 l}{2}\right) + c_0$



or since $\frac{dy}{dx} = 0$ for $x = l$, $c_0 = w\,l^3/3$

Integrating again, $EIy = w\left(\frac{x^4}{24} - \frac{x^3 l}{6}\right) + \frac{wl^3 x}{3} + c_1$

As $y = 0$ for $x = 0$, $c_1 = 0$. From this we find

$$\lambda = y\,(x = l) = \frac{5\,w\,l^4}{24}\Big/ EI = \frac{5\,\rho g\,l^4}{2\,E\,h^2}$$

1.304 The deflection of the plate can be noticed by going to a co- rotating frame. In this frame each element of the plate experiences a pseudo force proportional to its mass. These forces have a moment which constitutes the bending moment of the problem. To calculate this moment we note that the acceleration of an element at a distance ξ from the axis is $a = \xi\,\beta$ and the moment of the forces exerted by the section between x and l is

$$N = \rho\, l\, h\, \beta \int_x^l \xi^2 \, d\xi = \frac{1}{3} \rho\, l\, h\, \beta \,(l^3 - x^3).$$

From the fundamental equation

$$E I \frac{d^2 y}{dx^2} = \frac{1}{3} \rho\, l\, h\, \beta\, (l^3 - x^3).$$

The moment of inertia $I = \int_{-h/2}^{+h/2} z^2\, l\, dz = \frac{l h^3}{12.}$

Note that the neutral surface (i.e. the surface which contains lines which are neither stretched nor compressed) is a vertical plane here and z is perpendicular to it.

$$\frac{d^2 y}{dx^2} = \frac{4 \rho \beta}{E h^2} (l^3 - x^3). \text{ Integrating}$$

$$\frac{dy}{dx} = \frac{4 \rho \beta}{E h^2} \left(l^3 x - \frac{x^4}{4} \right) + c_1$$

Since $\frac{dy}{dx} = 0$, for $x = 0$, $c_1 = 0$. Integrating again,

$$y = \frac{4 \rho \beta}{E h^2} \left(\frac{l^3 x^2}{2} - \frac{x^5}{20} \right) + c_2$$

$$c_2 = 0 \text{ because } y = 0 \text{ for } x = 0$$

Thus $$\lambda = y\,(x = l) = \frac{9 \rho \beta l^5}{5 E h^2}$$

1.305 (a) Consider a hollow cylinder of length l, outer radius $r + \Delta r$ inner radius r, fixed at one end and twisted at the other by means of a couple of moment N. The angular displacement φ, at a distance l from the fixed end, is proportional to both l and N. Consider an element of length dx at the twisted end. It is moved by an angle φ as shown. A vertical section is also shown and the twisting of the parallelopipe of length l and area $\Delta r\, dx$ under the action of the twisting couple can be discussed by elementary means. If f is the tangential force generated then shearing stress is $f / \Delta r\, dx$ and this must equal

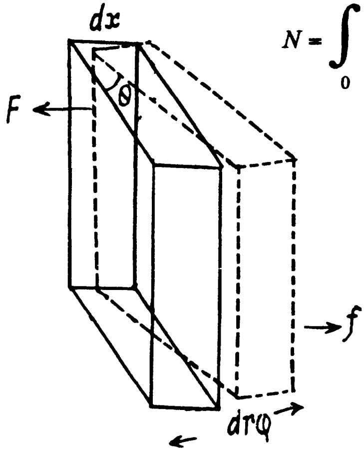
$$G \theta = G \frac{r \varphi}{l}, \text{ since } \theta = \frac{r \varphi}{l}.$$

Hence, $$f = G\, \Delta r\, dx \frac{r \varphi}{l}.$$

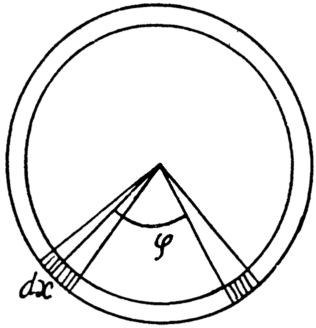
The force f has moment fr about the axis and so the total moment is

$$N = G\, \Delta r \frac{\varphi}{l} r^2 \int dx = \frac{2 \pi r^3 \Delta r\, \varphi}{l} G$$

(b) For a solid cylinder we must integrate over r. Thus



$$N = \int_0^r \frac{2\pi r^3\, dr\, \varphi\, G}{l} = \frac{\pi r^4 G \varphi}{2l}$$



1.306 Clearly $N = \int_{d_1/2}^{d_2/2} \frac{2\pi r^3\, d r\, \varphi\, G}{l} = \frac{\pi}{32\, l} G \varphi (d_2^4 - d_1^4)$

using
$$G = 81 \text{ G Pa} = 8{\cdot}1 \times 10^{10} \frac{\text{N}}{\text{m}^2}$$

$$d_2 = 5 \times 10^{-2} \text{ m}, \; d_1 = 3 \times 10^{-2} \text{ m}$$

$$\varphi = 2{\cdot}0° = \frac{\pi}{90} \text{ radians}, \; l = 3 \text{ m}$$

$$N = \frac{\pi \times 8{\cdot}1 \times \pi}{32 \times 3 \times 90} (625 - 81) \times 10^2 \text{ N·m}$$

$$= 0{\cdot}5033 \times 10^3 \text{ N·m} \approx 0{\cdot}5 \text{ k N·m}$$

1.307 The maximum power that can be transmitted by means of a shaft rotating about its axis is clearly $N\omega$ where N is the moment of the couple producing the maximum permissible torsion, φ. Thus

$$P = \frac{\pi r^4 G \varphi}{2l} \cdot \omega = 16{\cdot}9 \text{ kw}$$

1.308 Consider an elementary ring of width dr at a distant r from the axis. The part outside exerts a couple $N + \frac{dN}{dr} dr$ on this ring while the part inside exerts a couple N in the opposite direction. We have for equilibrium

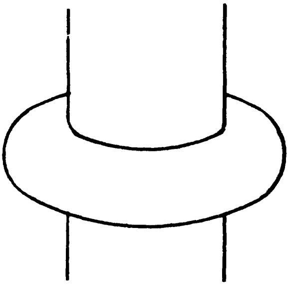
$$\frac{dN}{dr} dr = - dI\beta$$

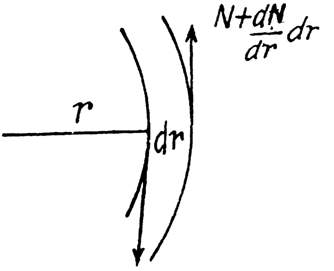
where dI is the moment of inertia of the elementary ring, β is the angular acceleration and minus sign is needed because the couple $N(r)$ decreases, with distance vanshing at the outer radius, $N(r_2) = 0$. Now

$$dI = \frac{m}{\pi(r_2^2 - r_1^2)} 2\pi r\,dr\,r^2$$

Thus $$dN = \frac{2m\beta}{(r_1^2 - r_1^2)} r^3\,dr$$

or, $$N = \frac{1}{2}\frac{m\beta}{(r_2^2 - r_1^2)}(r_2^4 - r^4), \text{ on integration}$$





1.309 We assume that the deformation is wholly due to external load, neglecting the effect of the weight of the rod (see next problem). Then a well known formula says,

elastic energy per unit volume

$$= \frac{1}{2}\text{stress} \times \text{strain} = \frac{1}{2}\sigma\varepsilon$$

This gives $\frac{1}{2}\frac{m}{\rho}E\varepsilon^2 \approx 0{\cdot}04$ kJ for the total deformation energy.

1.310 When a rod is deformed by its own weight the stress increases as one moves up, the stretching force being the weight of the portion below the element considered.

The stress on the element dx is

$$\rho\pi r^2(l-x)g/\pi r^2 = \rho g(l-x)$$

The extension of the element is

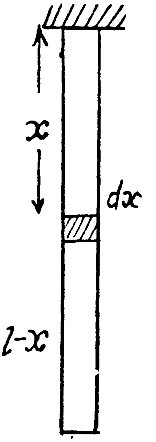
$$\Delta\,dx = d\,\Delta x = \rho g(l-x)\,dx/E$$

Integrating $\Delta l = \frac{1}{2}\rho g l^2/E$ is the extension of the whole rod. The elastic energy of the element is

$$\frac{1}{2}\rho g(l-x)\frac{\rho g(l-x)}{E}\pi r^2\,dx$$

Integrating

$$\Delta U = \frac{1}{6}\pi r^2\rho^2 g^2 l^3/E = \frac{2}{3}\pi r^2 l E\left(\frac{\Delta l}{l}\right)^2$$



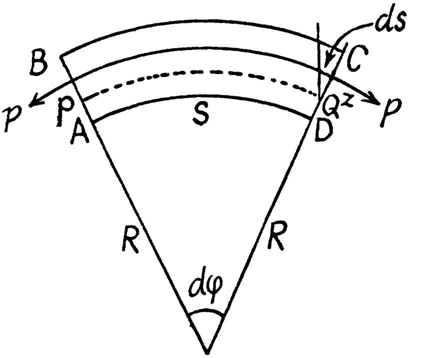
1.311 The work done to make a loop out of a steel band appears as the elastic energy of the loop and may be calculated from the same.

If the length of the band is l, the radius of the loop $R = \frac{l}{2\pi}$. Now consider an element $ABCD$ of the loop. The elastic energy of this element can be calculated by the same sort of arguments as used to derive the formula for internal bending moment. Consider a fibre at a distance z from the neutral surface PQ. This fibre experiences a force p and undergoes an extension ds where $ds = Z\,d\varphi$, while $PQ = s = R\,d\varphi$. Thus strain $\frac{ds}{s} = \frac{Z}{R}$. If α is the cross sectional area of the fibre, the elastic energy associated with it is

$$\frac{1}{2}E\left(\frac{Z}{R}\right)^2 R\,d\varphi\,\alpha$$

Summing over all the fibres we get

$$\frac{E\,I\,\varphi}{2R}\sum \alpha Z^2 = \frac{E\,I\,d\varphi}{2R}$$



For the whole loop this gives,

using $\int d\varphi = 2\pi$,

$$\frac{E\,I\,\pi}{R} = \frac{2\,E\,I\,\pi^2}{l}$$

Now $$I = \int_{-\delta/2}^{\delta/2} Z^2\,h\,dZ = \frac{h\,\delta^3}{12}$$

So the energy is $$\frac{1}{6}\frac{\pi^2 E\,h\,\delta^3}{l} = 0{\cdot}08\text{ kJ}$$

1.312 When the rod is twisted through an angle θ, a couple

$N(\theta) = \frac{\pi r^4 G}{2l}\theta$ appears to resist this. Work done in twisting the rod by an angle φ is then

$$\int_0^{\varphi} N(\theta)\,d\theta = \frac{\pi r^4 G}{4l}\varphi^2 = 7\text{ J} \quad \text{on putting the values.}$$

1.313 The energy between radii r and $r + dr$ is, by differentiation, $\frac{\pi r^3\,dr}{l}G\,\varphi^2$

Its density is $$\frac{\pi r^3\,dr}{2\pi r\,dr\,l}\frac{G\,\varphi^2}{l} = \frac{1}{2}\frac{G\,\varphi^2 r^2}{l^2}$$

1.314 The energy density is as usual 1/2 stress × strain. Stress is the pressure $\rho g h$. Strain is $\beta \times \rho g h$ by defination of β. Thus

$$u = \frac{1}{2}\beta(\rho g h)^2 = 23{\cdot}5\text{ kJ/m}^3 \quad \text{on putting the values.}$$

1.7 HYDRODYNAMICS

1.315 Between 1 and 2 fluid particles are in nearly circular motion and therefore have centripetal acceleration. The force for this acceleration, like for any other situation in an ideal fluid, can only come from the pressure variation along the line joining 1 and 2. This requires that pressure at 1 should be greater than the pressure at 2 i.e.

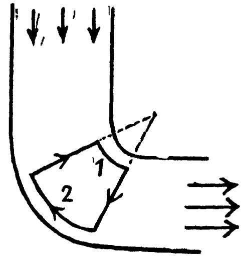
$$p_1 > p_2$$

so that the fluid particles can have required acceleration. If there is no turbulence, the motion can be taken as irrotational. Then by considering

$$\oint \vec{v} \cdot d\vec{l} = 0$$

along the circuit shown we infer that

$$v_2 > v_1$$



(The portion of the circuit near 1 and 2 are streamlines while the other two arms are at right angle to streamlines)

In an incompressible liquid we also have div $\vec{v} = 0$

By electrostatic analogy we then find that the density of streamlines is proportional to the velocity at that point.

1.316 From the conservation of mass

$$v_1 S_1 = v_2 S_2 \qquad (1)$$

But $S_1 < S_2$ as shown in the figure of the problem, therefore

$$v_1 > v_2$$

As every streamline is horizontal between 1 & 2, Bernoull's theorem becomes

$$p + \frac{1}{2} \rho v^2 = \text{constant, which gives}$$

$$p_1 < p_2 \text{ as } v_1 > v_2$$

As the difference in height of the water column is Δh, therefore

$$p_2 - p_1 = \rho g \Delta h \qquad (2)$$

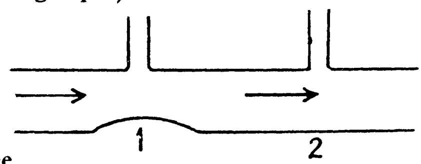
From Bernoull's theorem between points 1 and 2 of a streamline

$$p_1 + \frac{1}{2} \rho v_1^2 = p_2 + \frac{1}{2} \rho v_2^2$$

or,
$$p_2 - p_1 = \frac{1}{2} \rho (v_1^2 - v_2^2)$$

or
$$\rho g \Delta h = \frac{1}{2} \rho (v_1^2 - v_2^2) \qquad (3) \text{ (using Eq. 2)}$$

using (1) in (3), we get

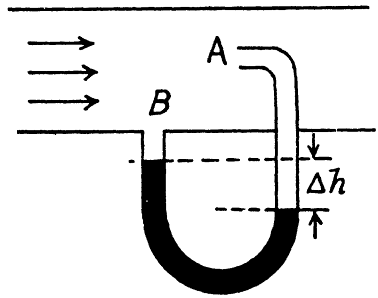


$$v_1 = S_2 \sqrt{\frac{2 g \Delta h}{S_2^2 - S_1^2}}$$

Hence the sought volume of water flowing per see

$$Q = v_1 S_1 = S_1 S_2 \sqrt{\frac{2 g \Delta h}{S_2^2 - S_1^2}}$$

1.317 Applying Bernoulli's theorem for the point A and B,



$$p_A = p_B + \frac{1}{2}\rho v^2 \quad \text{as,} \quad v_A = 0$$

or, $$\frac{1}{2}\rho v^2 = p_A - p_B = \Delta h \rho_0 g$$

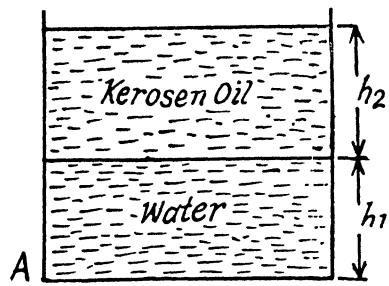
So, $$v = \sqrt{\frac{2\Delta h \rho_0 g}{\rho}}$$

Thus, rate of flow of gas, $Q = S v = S\sqrt{\dfrac{2\Delta h \rho_0 g}{\rho}}$

The gas flows over the tube past it at B. But at A the gas becomes stationary as the gas will move into the tube which already contains gas.

In applying Bernoulli's theorem we should remember that $\dfrac{p}{\rho} + \dfrac{1}{2}v^2 + gz$ is constant along a streamline. In the present case, we are really applying Bernoulli's theorem somewhat indirectly. The streamline at A is not the streamline at B. Nevertheless the result is correct. To be convinced of this, we need only apply Bernoull's theorem to the streamline that goes through A by comparing the situation at A with that above B on the same level. In steady conditions, this agrees with the result derived because there cannot be a transverse pressure differential.

1.318 Since, the density of water is greater than that of kerosene oil, it will collect at the bottom. Now, pressure due to water level equals $h_1 \rho_1 g$ and pressure due to kerosene oil level equals $h_2 \rho_2 g$. So, net pressure becomes $h_1 \rho_1 g + h_2 \rho_2 g$.

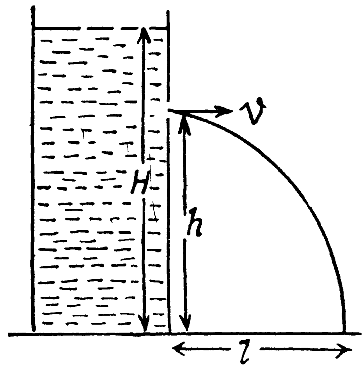


From Bernoulli's theorem, this pressure energy will be converted into kinetic energy while flowing through the whole A.

i.e. $$h_1 \rho_1 g + h_2 \rho_2 g = \frac{1}{2}\rho_1 v^2$$

Hence $$v = \sqrt{2\left(h_1 + h_2 \frac{\rho_2}{\rho_1}\right) g} = 3 \text{ m/s}$$

1.319 Let, H be the total height of water column and the hole is made at a height h from the bottom.



Then from Bernoulli's theorem

$$\frac{1}{2}\rho v^2 = (H - h)\rho g$$

or, $v = \sqrt{(H-h)\,2g}$, which is directed horizontally.

For the horizontal range, $l = v\ t$

$$= \sqrt{2g(H-h)} \cdot \sqrt{\frac{2h}{g}} = 2\sqrt{(Hh - h^2)}$$

Now, for maximum l, $\frac{d\,(Hh - h^2)}{dh} = 0$

which yields $$h = \frac{H}{2} = 25 \text{ cm.}$$

1.320 Let the velocity of the water jet, near the orifice be v', then applying Bernoullis theorem,

$$\frac{1}{2}\rho\, v^2 = h_0\, \rho\, g + \frac{1}{2}\rho\, v'^2$$

or, $$v' = \sqrt{v^2 - 2g\, h_0} \qquad (1)$$

Here the pressure term on both sides is the same and equal to atmospheric pressure. (In the problem book Fig. should be more clear.)

Now, if it rises upto a height h, then at this height, whole of its kinetic energy will be converted into potential energy. So,

$$\frac{1}{2}\rho\, v'^2 = \rho\, gh \quad \text{or} \quad h = \frac{v'^2}{2g}$$

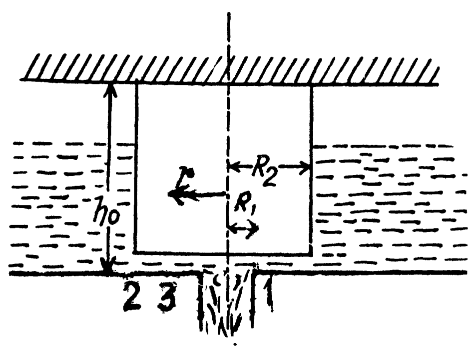
$$= \frac{v^2}{2g} - h_0 = 20 \text{ cm, [using Eq. (1)]}$$

1.321 Water flows through the small clearance into the orifice. Let d be the clearance. Then from the equation of continuity

$$(2\pi R_1 d)\, v_1 = (2\pi r\, d)\, v = (2\pi R_2 d)\, v_2$$

or $$v_1 R_1 = v\, r = v_2 R_2 \qquad (1)$$

where v_1, v_2 and v are respectively the inward radial velocities of the fluid at 1, 2 and 3.



Now by Bernoulli's theorem just before 2 and just after it in the clearance

$$p_0 + h\,\rho g = p_2 + \frac{1}{2}\rho v_2^2 \qquad (2)$$

Applying the same theorem at 3 and 1 we find that this also equals

$$p + \frac{1}{2}\rho v^2 = p_0 + \frac{1}{2}\rho v_1^2 \qquad (3)$$

(since the pressure in the orifice is p_0)

From Eqs. (2) and (3) we also hence

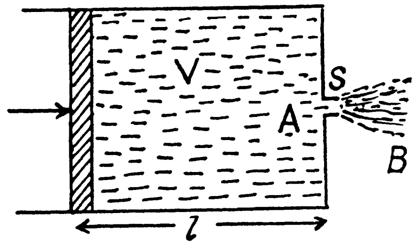
$$v_1 = \sqrt{2gh} \qquad (4)$$

and $$p = p_0 + \frac{1}{2}\rho v_1^2 \left(1 - \left(\frac{v}{v_1}\right)^2\right)$$

$$= p_0 + h\rho g \left(1 - \left(\frac{R_1}{r}\right)^2\right) \qquad \text{[Using (1) and (4)]}$$

1.322 Let the force acting on the piston be F and the length of the cylinder be l.

Then, work done $= Fl$ (1)



Applying Bernoulli's theorem for points A and B, $p = \frac{1}{2}\rho v^2$ where ρ is the density and v is the velocity at point B. Now, force on the piston,

$$F = pA = \frac{1}{2}\rho v^2 A \qquad (2)$$

where A is the cross section area of piston.

Also, discharge through the orifice during time interval $t = Svt$ and this is equal to the volume of the cylinder, i.e.,

$$V = Svt \quad \text{or} \quad v = \frac{V}{St} \qquad (3)$$

From Eq. (1), (2) and (3) work done

$$= \frac{1}{2}\rho v^2 A\, l = \frac{1}{2}\rho A \frac{V^2}{(St)^2} l = \frac{1}{2}\rho V^3/S^3 t^2 \quad (\text{as } Al = V)$$

1.323 Let at any moment of time, water level in the vessel be H then speed of flow of water through the orifice, at that moment will be

$$v = \sqrt{2gH} \qquad (1)$$

In the time interval dt, the volume of water ejected through orifice,

$$dV = s\,v\,dt \qquad (2)$$

On the other hand, the volume of water in the vessel at time t equals

$$V = S\,H$$

Differentiating (3) with respect to time,

$$\frac{dV}{dt} = S\frac{dH}{dt} \quad \text{or} \quad dV = S\,dH \qquad (4)$$

Eqs. (2) and (4)

$$S\,dH = s\,v\,dt \quad \text{or} \quad dt = \frac{S}{s}\frac{dH}{\sqrt{2gH}}, \text{from (2)}$$

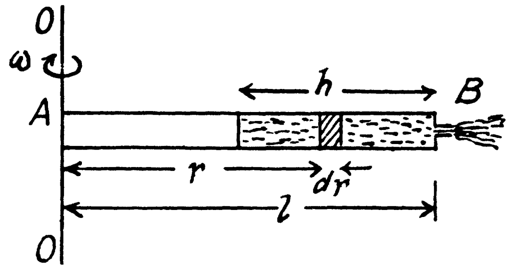
Integrating, $$\int_0^t dt = \frac{S}{s\sqrt{2h}}\int_h^0 \frac{dh}{\sqrt{H}}$$

Thus, $$t = \frac{S}{s}\sqrt{\frac{2h}{g}}$$

1.324 In a rotating frame (with constant angular velocity) the Eulerian equation is

$$-\vec{\nabla} p + \rho \vec{g} + 2\rho(\vec{v}' \times \vec{\omega}) + \rho\omega^2 \vec{r} = \rho\frac{d\vec{v}'}{dt}$$

In the frame of rotating tube the liquid in the "column" is practically static because the orifice is sufficiently small. Thus the Eulerian Eq. in projection form along $\vec{r}$ (which is the position vector of an arbitrary liquid element of lenth dr relative to the rotation axis) reduces to



$$\frac{-dp}{dr} + \rho\,\omega^2 r = 0$$

or, $$dp = \rho\,\omega^2 r\,dr$$

so, $$\int_{p_0}^{p} dp = \rho\omega^2 \int_{(l-h)}^{r} r\,dr$$

Thus $$p(r) = p_0 + \frac{\rho\omega^2}{2}\left[r^2 - (l-h)^2\right] \tag{1}$$

Hence the pressure at the end B just before the orifice i.e.

$$p(l) = p_0 + \frac{\rho\,\omega^2}{2}(2\,l\,h - h^2) \tag{2}$$

Then applying Bernoull's theorem at the orifice for the points just inside and outside of the end B

$$p_0 + \frac{1}{2}\rho\,\omega^2 (2\,l\,h - h^2) = p_0 + \frac{1}{2}\rho\,v^2$$ (where v is the sought velocity)

So, $$v = \omega h\sqrt{\frac{2l}{h} - 1}$$

1.325 The Euler's equation is $\rho\frac{d\vec{v}}{dt} = \vec{f} - \vec{\nabla}p = -\vec{\nabla}(p + \rho\,gz)$, where z is vertically upwards.

Now $$\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v} \tag{1}$$

But $$(\vec{v}\cdot\vec{\nabla})\,\vec{v} = \vec{\nabla}\left(\frac{1}{2}v^2\right) - \vec{v}\times \text{Curl}\,\vec{v} \tag{2}$$

we consider the steady (i.e. $\partial\vec{v}/\partial t = 0$) flow of an incompressible fluid then ρ = constant.
and as the motion is irrotational Curl $\vec{v} = 0$

So from (1) and (2) $$\rho\,\vec{\nabla}\left(\frac{1}{2}v^2\right) = -\vec{\nabla}(p + \rho\,gz)$$

or, $$\vec{\nabla}\left(p + \frac{1}{2}\rho\,v^2 + \rho gz\right) = 0$$

Hence $$p + \frac{1}{2}\rho\,v^2 + \rho\,gz = \text{constant.}$$

1.326 Let the velocity of water, flowing through A be v_A and that through B be v_B, then discharging rate through $A = Q_A = S\,v_A$ and similarly through $B = S\,v_B$.

Now, force of reaction at A,

$$F_A = \rho\,Q_A\,v_A = \rho\,S\,v_B^2$$

Hence, the net force,

$$F = \rho S (v_A^2 - v_B^2) \quad \text{as} \quad \vec{F}_A \uparrow\downarrow \vec{F}_B \qquad (1)$$

Applying Bernoulli's theorem to the liquid flowing out of A we get

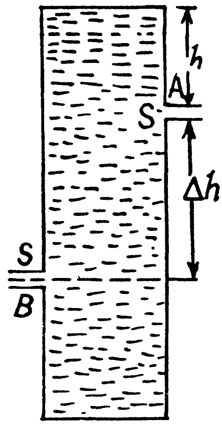
$$p_0 + \rho g h = p_0 + \frac{1}{2}\rho v_A^2$$

and similarly at B

$$p_0 + \rho g (h + \Delta h) = p_0 + \frac{1}{2}\rho v_B^2$$

Hence $$(v_B^2 - v_A^2)\frac{\rho}{2} = \Delta h \rho g$$

Thus $$F = 2\rho g S \Delta h = 0{\cdot}50\ \text{N}$$



1.327 Consider an element of height dy at a distance y from the top. The velocity of the fluid coming out of the element is

$$v = \sqrt{2gy}$$

The force of reaction dF due to this is $dF = \rho\, dA\, v^2$, as in the previous problem, $= \rho\,(b\,dy)\,2gy$

Integrating $$F = \rho g b \int_{h-1}^{h} 2y\, dy$$

$$= \rho g b\,[h^2 - (h-l)^2] = \rho g b l\,(2h - l)$$

(The slit runs from a depth $h - l$ to a depth h from the top.)

1.328 Let the velocity of water flowing through the tube at a certain instant of time be u, then $u = \dfrac{Q}{\pi r^2}$, where Q is the rate of flow of water and πr^2 is the cross section area of the tube.

From impulse momentum theorem, for the stream of water striking the tube corner, in x–direction in the time interval dt,

$$F_x\, dt = -\rho Q u\, dt \quad \text{or} \quad F_x = -\rho Q u$$

and similarly, $F_y = \rho Q u$

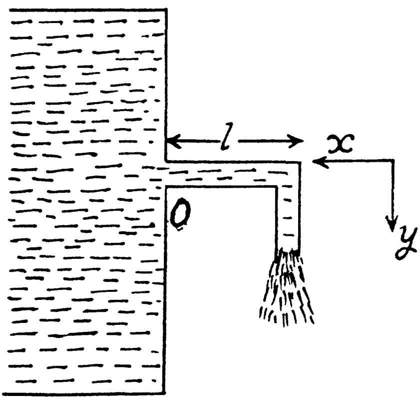
Therefore, the force exerted on the water stream by the tube,

$$\vec{F} = -\rho Q u\,\vec{i} + \rho Q u\,\vec{j}$$

According to third law, the reaction force on the tube's wall by the stream equals $(-\vec{F})$

$$= \rho Q u\,\vec{i} - \rho Q u\,\vec{j}.$$

Hence, the sought moment of force about 0 becomes



$$\vec{N} = l(-\vec{i}) \times (\rho Q u\,\vec{i} - \rho Q u\,\vec{j}) = \rho Q u\, l\,\vec{k} = \frac{\rho Q^2}{\pi r^2}\, l\,\vec{k}$$

and $$|\vec{N}| = \frac{\rho Q^2 l}{\pi r^2} = 0{\cdot}70\ \text{N·m}$$

1.329 Suppose the radius at A is R and it decreases uniformaly to r at B where $S = \pi R^2$ and $s = \pi r^2$. Assume also that the semi vectical angle at 0 is α. Then

$$\frac{R}{L_2} = \frac{r}{L_1} = \frac{y}{x}$$

So
$$y = r + \frac{R-r}{L_2 - L_1}(x - L_1)$$

where y is the radius at the point P distant x from the vertex O. Suppose the velocity with which the liquid flows out is V at A, v at B and u at P. Then by the equation of continuity

$$\pi R^2 V = \pi r^2 v = \pi y^2 u$$

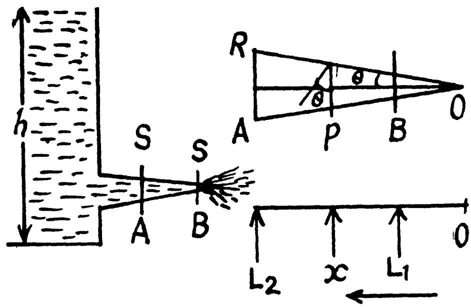
The velocity v of efflux is given by

$$v = \sqrt{2gh}$$

and Bernoulli's theorem gives

$$p_p + \frac{1}{2}\rho u^2 = p_0 + \frac{1}{2}\rho v^2$$

where p_p is the pressure at P and p_0 is the atmospheric pressure which is the pressure just outside of B. The force on the nozzle tending to pull it out is then



$$F = \int (p_p - p_0)\sin\theta\, 2\pi y\, ds$$

We have subtracted p_0 which is the force due to atmosphenic pressure the factor $\sin\theta$ gives horizontal component of the force and ds is the length of the element of nozzle surface, $ds = dx \sec\theta$ and

$$\tan\theta = \frac{R-r}{L_2 - L_1}$$

Thus

$$F = \int_{L_1}^{L_2} \frac{1}{2}(v^2 - u^2)\,\rho\, 2\pi y \frac{R-r}{L_2-L_1}\,dx$$

$$= \pi\rho \int_r^R v^2\left(1 - \frac{r^4}{y^4}\right) y\, dy$$

$$= \pi\rho v^2 \frac{1}{2}\left(R^2 - r^2 + \frac{r^4}{R^2} - r^2\right) = \rho g h \left(\frac{\pi(R^2 - r^2)^2}{R^2}\right)$$

$$= \rho g h\ (S-s)^2/S = 6{\cdot}02\ \text{N} \quad \text{on putting the values.}$$

Note : If we try to calculate F from the momentum change of the liquid flowing out w will be wrong even as regards the sign of the force.

There is of course the effect of pressure at S and s but quantitative derivation of F fron Newton's law is difficult.

1.330 The Euler's equation is $\rho \frac{d\vec{v}}{dt} = \vec{f} - \vec{\nabla} p$ in the space fixed frame where $\vec{f} = -\rho g \vec{k}$ downward. We assume incompressible fluid so ρ is constant.

Then $\vec{f} = -\vec{\nabla}(\rho g z)$ where z is the height vertically upwards from some fixed origin. We go to rotating frame where the equation becomes

$$\rho \frac{d\vec{v}'}{dt} = -\vec{\nabla}(p + \rho g z) + \rho \omega^2 \vec{r} + 2\rho (\vec{v}' \times \vec{\omega})$$

the additional terms on the right are the well known coriolis and centrifugal forces. In the frame rotating with the liquid $\vec{v}' = 0$ so

$$\vec{\nabla}\left(p + \rho g z - \frac{1}{2}\rho \omega^2 r^2\right) = 0$$

or
$$p + \rho g z - \frac{1}{2}\rho \omega^2 r^2 = \text{constant}$$

On the free surface $p =$ constant, thus

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}$$

If we choose the origin at point $r = 0$ (i.e. the axis) of the free surface then "cosntant" = 0 and

$$z = \frac{\omega^2}{2g} r^2 \quad \text{(The paraboloid of revolution)}$$

At the bottom $z =$ constant

So
$$p = \frac{1}{2}\rho \omega^2 r^2 + \text{constant}$$

If $p = p_0$ on the axis at the bottom, then

$$p = p_0 + \frac{1}{2}\rho \omega^2 r^2.$$

1.331 When the disc rotates the fuild in contact with, corotates but the fluid in contact with the walls of the cavity does not rotate. A velocity gradient is then set up leading to viscous forces. At a distance r from the axis the linear velocity is ωr so there is a velocity gradient $\frac{\omega r}{h}$ both in the upper and lower clearance. The corresponding force on the element whose radial width is dr is

$$\eta\, 2\pi r dr\, \frac{\omega r}{h} \quad \text{(from the formular } F = \eta A \frac{dv}{dx}\text{)}$$

The torque due to this force is

$$\eta\, 2\pi r dr\, \frac{\omega r}{h}\, r$$

and the net torque considering both the upper and lower clearance is

$$2\int_0^R \eta 2\pi r^3\, dr \frac{\omega}{h}$$

$$= \pi R^4 \omega \eta / h$$

So power developed is

$$P = \pi R^4 \omega^2 \eta / h = 9 \cdot 05 \text{ W (on putting the values).}$$

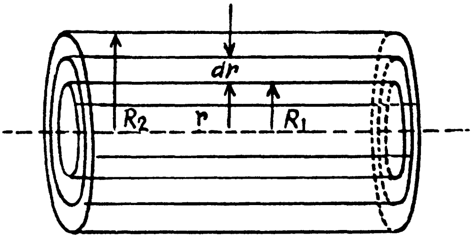
(As instructed end effects i.e. rotation of fluid in the clearance $r > R$ has been neglected.)

1.332 Let us consider a coaxial cylinder of radius r and thickness dr, then force of friction or viscous force on this elemental layer, $F = 2\pi r l \eta \frac{dv}{dr}$.

This force must be constant from layer to layer so that steady motion may be possible.

or, $$\frac{F\,dr}{r} = 2\pi l \eta\, dv. \qquad (1)$$

Integrating,



$$F \int_{R_2}^{r} \frac{dr}{r} = 2\pi l \eta \int_0^v dv$$

or, $$F \ln\left(\frac{r}{R_2}\right) = 2\pi l \eta v \qquad (2)$$

Putting $r = R_1$, we get

$$F \ln \frac{R_1}{R_2} = 2\pi l \eta v_0$$

From (2) by (3) we get,

$$v = v_0 \frac{\ln r/R_2}{\ln R_1/R_2}$$

Note : The force F is supplied by the agency which tries to carry the inner cylinder with velocity v_0 .

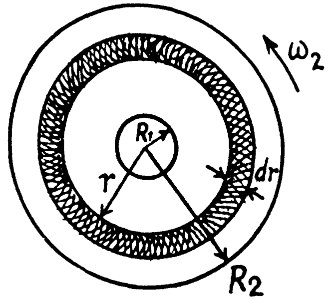
1.333 (a) Let us consider an elemental cylinder of radius r and thickness dr then from Newton's formula

$$F = 2\pi r l \eta r \frac{d\omega}{dr} = 2\pi l \eta r^2 \frac{d\omega}{dr}$$

and moment of this force acting on the element,

$$N = 2\pi r^2 l \eta \frac{d\omega}{dr} r = 2\pi r^3 l \eta \frac{d\omega}{dr}$$

or, $$2\pi l \eta\, d\omega = N \frac{dr}{r^3} \qquad (2)$$



As in the previous problem N is constant when conditions are steady

Integrating, $$2\pi l \eta \int_0^{\omega} d\omega = N \int_{R_1}^{r} \frac{dr}{r^3}$$

or, $$2\pi l \eta \omega = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right] \qquad (3)$$

Putting $r = R_2$, $\omega = \omega_2$, we get

$$2\pi l \eta \omega_2 = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right] \qquad (4)$$

From (3) and (4),

$$\omega = \omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right]$$

(b) From Eq. (4),

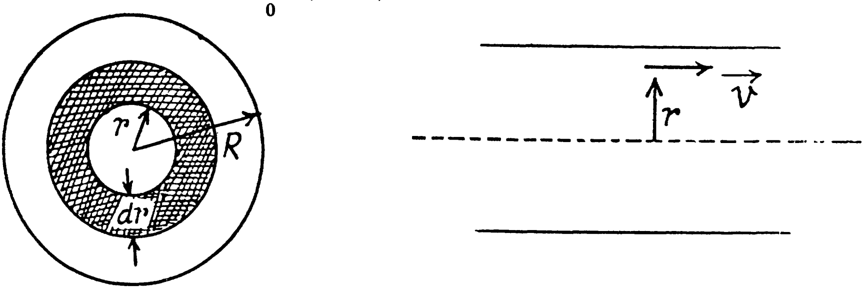
$$N_1 = \frac{N}{l} = 4\pi\eta\,\omega_2 \frac{R_1^2 R_2^2}{R_2^2 - R_1^2}$$

1.334 (a) Let dV be the volume flowing per second through the cylindrical shell of thickness dr then,

$$dV = -(2\pi r\,dr)\,v_0\left(1 - \frac{r^2}{R^2}\right) = 2\pi v_0\left(r - \frac{r^3}{R^2}\right)dr$$

and the total volume,

$$V = 2\pi v_0 \int_0^R \left(r - \frac{r^3}{R^2}\right)dr = 2\pi v_0 \frac{R^2}{4} = \frac{\pi}{2}R^2 v_0$$



(b) Let, dE be the kinetic energy, within the above cylindrical shell. Then

$$dT = \frac{1}{2}(dm)\,v^2 = \frac{1}{2}(2\pi\,r\,l\,d\,r\rho)\,v^2$$

$$= \frac{1}{2}(2\pi\,l\,\rho)\,r\,dr\,v_0^2\left(1 - \frac{r^2}{R^2}\right) = \pi\,l\,\rho\,v_0\left[r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right]dr$$

Hence, total energy of the fluid,

$$T = \pi\,l\,\rho\,v_0^2 \int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right)dr = \frac{\pi R^2 \rho\,l\,v_0^2}{6}$$

(c) Here frictional force is the shearing force on the tube, exerted by the fluid, which equals $-\eta S \frac{dv}{dt}$.

Given, $$v = v_0\left(1 - \frac{r^2}{R^2}\right)$$

So, $$\frac{dv}{dr} = -2v_0\frac{r}{R^2}$$

And at $$r = R,\ \frac{dv}{dr} = -\frac{2v_0}{R}$$

Then, viscous force is given by, $F = -\eta\,(2\pi Rl)\left(\frac{dv}{dr}\right)_{r=R}$

$$= -2\pi R\eta\, l\left(-\frac{2v_0}{R}\right) = 4\pi\eta\, v_0\, l$$

(d) Taking a cylindrical shell of thickness dr and radius r viscous force,

$$F = -\eta\,(2\pi rl)\frac{dv}{dr},$$

Let Δp be the pressure difference, then net force on the element $= \Delta p\,\pi r^2 + 2\pi\eta\, l\, r\frac{dv}{dr}$

But, since the flow is steady, $F_{net} = 0$

or, $$\Delta p = \frac{-2\pi l\eta\, r\frac{dv}{dr}}{\pi r^2} = \frac{-2\pi l\,\eta\, r\left(-2v_0\frac{r}{R^2}\right)}{\pi r^2} = 4\eta\, v_0\, l/R^2$$

1.335 The loss of pressure head in travelling a distance l is seen from the middle section to be $h_2 - h_1 = 10$ cm. Since $h_2 - h_1 = h_1$ in our problem and $h_3 - h_2 = 15\text{ cm} = 5 + h_2 - h_1$, we see that a pressure head of 5 cm remains incompensated and must be converted into kinetic energy, the liquid flowing out. Thus

$$\frac{\rho v^2}{2} = \rho g\,\Delta h \quad \text{where } \Delta h = h_3 - h_2$$

Thus $$v = \sqrt{2g\Delta h} = 1\text{ m/s}$$

1.336 We know that, Reynold's number (R_e) is defined as, $R_e = \rho\, v\, l/\eta$, where v is the velocity l is the characteristic length and η the coefficient of viscosity. In the case of circular cross section the chracteristic length is the diameter of cross-section d, and v is taken as average velocity of flow of liquid.

Now, R_{e_1} (Reynold's number at x_1 from the pipe end) $= \frac{\rho d_1 v_1}{m\,\eta}$ where v_1 is the velocity at distance x_1

and similarly, $$R_{e_2} = \frac{\rho\, d_2\, v_2}{\eta} \quad \text{so} \quad \frac{R_{e_1}}{R_{e_2}} = \frac{d_1 v_1}{d_2 v_2}$$

From equation of continuity, $A_1 v_1 = A_2 v_2$

or, $$\pi r_1^2 v_1 = \pi r_2^2 v_2 \quad \text{or} \quad d_1 v_1 r_1 = d_2 v_2 r_2$$

$$\frac{d_1 v_1}{d_2 v_2} = \frac{r_2}{r_1} = \frac{r_0 e^{-\alpha x_2}}{r_0 e^{-\alpha x_1}} = e^{-\alpha\Delta x} \quad (\text{as } x_2 - x_1 = \Delta x)$$

Thus $$\frac{R_{e_2}}{R_{e_1}} = e^{\alpha\Delta x} = 5$$

1.337 We know that Reynold's number for turbulent flow is greater than that on laminar flow.

Now, $$(R_e)_l = \frac{\rho v d}{\eta} = \frac{2\rho_1 v_1 r_1}{\eta_1} \quad \text{and} \quad (R_e)_t = \frac{2\rho_2 v_2 r_2}{\eta}$$

But, $(R_e)_t \geq (R_e)_l$

so $$v_{2_{max}} = \frac{\rho_1 v_1 r_1 \eta_2}{\rho_2 r_2 \eta_1} = 5\ \mu\text{ m/s on putting the values.}$$

1.338 We have $R = \frac{v \rho_0 d}{\eta}$ and v is given by

$$6 \pi \eta r v = \frac{4\pi}{3} r^2 (\rho - \rho_0) g$$

(ρ = density of lead, ρ_0 = density of glycerine.)

$$v = \frac{2}{9\eta}(\rho - \rho_0) g r^2 = \frac{1}{18 \eta}(\rho - \rho_0) g d^2$$

Thus $$\frac{1}{2} = \frac{1}{18 \eta^2}(\rho - \rho_o) g \rho_0 d^3$$

and $$d = [9 \eta^2 / \rho_0 (\rho - \rho_0) g]^{1/3} = 5{\cdot}2\text{ mm on putting the values.}$$

1.339 $$m \frac{dv}{dt} = mg - 6 \pi \eta r v$$

or $$\frac{dv}{dt} + \frac{6 \pi \eta r}{m} v = g$$

or $$\frac{dv}{dt} + kv = g, k = \frac{6 \pi \eta r}{m}$$

or $$e^{kt} \frac{dv}{dt} + k e^{kt} v = g e^{kt} \text{ or } \frac{d}{dt} e^{kt} v = g e^{kt}$$

or $$v e^{kt} = \frac{g}{k} e^{kt} + C \text{ or } v = \frac{g}{k} + C e^{-kt} \text{ (where } C \text{ is const.)}$$

Since $$v = 0 \text{ for } t = 0,\ 0 = \frac{g}{k} + C$$

So $$C = -\frac{g}{k}$$

Thus $$v = \frac{g}{k}(1 - e^{-kt})$$

The steady state velocity is $\frac{g}{k}$.

v differs from $\frac{g}{k}$ by n where $e^{-kt} = n$

or $$t = \frac{1}{k} \ln n$$

Thus $$\frac{1}{k} = -\frac{\frac{4\pi}{3} r^3 P}{6 \pi \eta r} = -\frac{4 r^2 \rho}{18 \eta} = -\frac{d^2 \rho}{18 \eta}$$

We have neglected buoyancy in olive oil.

1.8 RELATIVISTIC MECHANICS

1.340 From the formula for length contraction

$$\left(l_0 - l_0\sqrt{1-\frac{v^2}{c^2}}\right) = \eta\, l_0$$

So, $$1-\frac{v^2}{c^2} = (1-\eta)^2 \quad \text{or} \quad v = c\sqrt{\eta\,(2-\eta)}$$

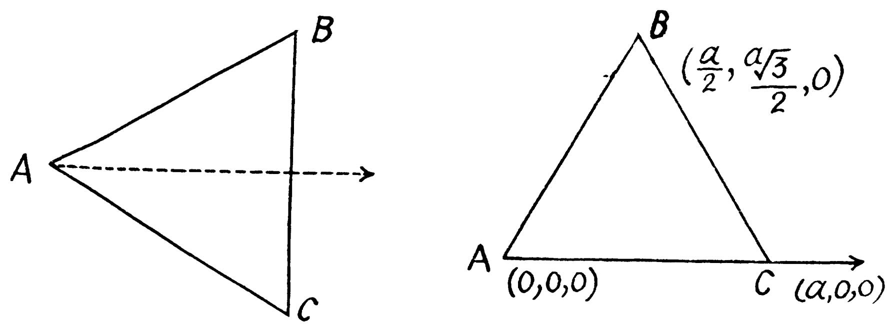
1.341 (a) In the frame in which the triangle is at rest the space coordinates of the vertices are (000), $\left(a\frac{\sqrt{3}}{2}, +\frac{a}{2}, 0\right)\left(a\frac{\sqrt{3}}{2}, -\frac{a}{2}, 0\right)$, all measured at the same time t. In the moving frame the corresponding coordinates at time t' are

$$A:(vt', 0, 0), B:\left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2}+vt', \frac{a}{2}, 0\right) \text{ and } C:\left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2}+vt', -\frac{a}{2}, 0\right)$$

The perimeter P is then

$$P = a + 2a\left(\frac{3}{4}(1-\beta^2)+\frac{1}{4}\right)^{1/2} = a\left(1+\sqrt{4-3\beta^2}\right)$$

(b) The coordinates in the first frame are shown at time t. The coordinates in the moving frame are,



$$A:(vt', 0,0), B:\left(\frac{a}{2}\sqrt{1-\beta^2}+vt', a\frac{\sqrt{3}}{2}, 0\right), C:\left(a\sqrt{1-\beta^2}+vt', 0, 0\right)$$

The perimeter P is then

$$P = a\sqrt{1-\beta^2}+\frac{a}{2}\left[1-\beta^2+3\right]^{1/2}\times 2 = a\left(\sqrt{1-\beta^2}+\sqrt{4-\beta^2}\right) \text{ here } \beta = \frac{V}{c}$$

1.342 In the rest frame, the coordinates of the ends of the rod in terms of proper length l_0

$$A:(0,0,0)\; B:(l_0\cos\theta_0, l_0\sin\theta_0, 0)$$

at time t. In the laboratory frame the coordinates at time t' are

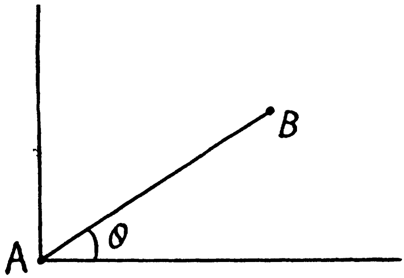
$$A:(vt', 0, 0), B:\left(l_0\cos\theta_0\sqrt{1-\beta^2}+vt', l_0\sin\theta_0, 0\right)$$

Therefore we can write,

$$l\cos\theta_0 = l_0\cos\theta_0\sqrt{1-\beta^2} \text{ and } l\sin\theta = l_0\sin\theta_0$$

Hence $l_0^2 = (l^2)\left(\frac{\cos^2\theta + (1-\beta^2)\sin^2\theta}{1-\beta^2}\right)$

or, $= \sqrt{\frac{1-\beta^2\sin^2\theta}{1-\beta^2}}$



1.343 In the frame K in which the cone is at rest the coordinates of A are (0,0,0) and of B are $(h, h\tan\theta, 0)$. In the frame K', which is moving with velocity v along the axis of the cone, the coordinates of A and B at time t' are

$A : (-vt', 0,0), B : \left(h\sqrt{1-\beta^2} - vt', h\tan\theta, 0\right)$

Thus the taper angle in the frame K' is

$$\tan\theta' = \frac{\tan\theta}{\sqrt{1-\beta^2}}\left(= \frac{y'_B - y'_A}{x'_B - x'_A}\right)$$

and the lateral surface area is,

$$S = \pi h'^2 \sec\theta' \tan\theta'$$

$$= \pi h^2(1-\beta^2)\frac{\tan\theta}{\sqrt{1-\beta^2}}\sqrt{1+\frac{\tan^2\theta}{1-\beta^2}} = S_0\sqrt{1-\beta^2\cos^2\theta}$$

Here $S_0 = \pi h^2 \sec\theta \tan\theta$ is the lateral surface area in the rest frame and

$$h' = h\sqrt{1-\beta^2}, \ \beta = v/c.$$

1.344 Because of time dilation, a moving clock reads less time. We write,

$$t - \Delta t = t\sqrt{1-\beta^2}, \ \beta = \frac{v}{c}$$

Thus, $$1 - \frac{2\Delta t}{t} + \left(\frac{\Delta t}{t}\right)^2 = 1-\beta^2$$

or, $$v = c\sqrt{\frac{\Delta t}{t}\left(2-\frac{\Delta t}{t}\right)}$$

1.345 In the frame K the length l of the rod is related to the time of flight Δt by

$$l = v\,\Delta t$$

In the reference frame fixed to the rod (frame K')the proper length l_0 of the rod is given by

$$l_0 = v\,\Delta t'$$

But $$l_0 = \frac{l}{\sqrt{1-\beta^2}} = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}, \ \beta = \frac{v}{c}$$

Thus, $$v\,\Delta t' = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}$$

So $$1-\beta^2 = \left(\frac{\Delta t}{\Delta t'}\right)^2 \quad \text{or} \quad v = c\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

and $$l_0 = c\sqrt{(\Delta t')^2-(\Delta t)^2} = c\,\Delta t'\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

1.346 The distance travelled in the laboratory frame of reference is $v\Delta t$ where v is the velocity of the particle. But by time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \quad \text{So} \quad v = c\sqrt{1-(\Delta t_0/\Delta t)^2}$$

Thus the distance traversed is

$$c\Delta t\sqrt{1-(\Delta t_0/\Delta t)^2}$$

1.347 (a) If τ_0 is the proper life time of the muon the life time in the moving frame is

$$\frac{\tau_0}{\sqrt{1-v^2/c^2}} \quad \text{and hence } l = \frac{v\,\tau_0}{\sqrt{1-v^2/c^2}}$$

Thus $$\tau_0 = \frac{l}{v}\sqrt{1-v^2/c^2}$$

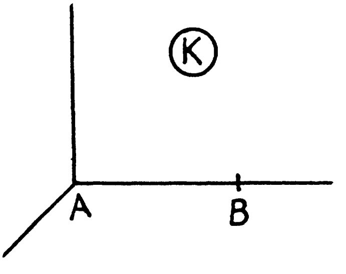
(The words "from the muon's stand point" are not part of any standard terminology)

1.348 In the frame K in which the particles are at rest, their positions are A and B whose coordinates may be taken as,

$$A:(0,0,0),\ B = (l_0, 0, 0)$$

In the frame K' with respect to which K is moving with a velocity v the coordinates of A and B at time t' in the moving frame are

$$A = (vt', 0,0)\ B = \left(l_0\sqrt{1-\beta^2}+vt', 0,0\right),\ \beta = \frac{v}{c}$$



Suppose B hits a stationary target in K' after time t'_B while A hits it after time $t_B + \Delta t$. Then,

$$l_0\sqrt{1-\beta^2} + vt'_B = v\,(t'_B + \Delta t)$$

So, $$l_0 \frac{v\,\Delta t}{\sqrt{1-v^2/c^2}}$$

1.349 In the reference frame fixed to the ruler the rod is moving with a velocity v and suffers Lorentz contraction. If l_0 is the proper length of the rod, its measured length will be

$$\Delta x_1 = l_0\sqrt{1-\beta^2},\ \beta = \frac{v}{c}$$

In the reference frame fixed to the rod the ruler suffers Lorentz contraction and we must have

$$\Delta x_2 \sqrt{1-\beta^2} = l_0 \text{ thus } l_0 = \sqrt{\Delta x_1 \Delta x_2}$$

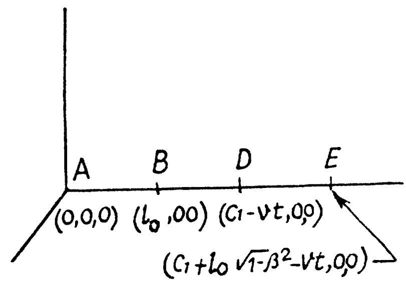
and
$$1-\beta^2 = \frac{\Delta x_1}{\Delta x_2} \text{ or } v = c\sqrt{1-\frac{\Delta x_1}{\Delta x_2}}$$

1.350 The coordinates of the ends of the rods in the frame fixed to the left rod are shown. The points B and D coincides when

$$l_0 = c_1 - vt_0 \text{ or } t_0 = \frac{c_1 - l_0}{v}$$

The points A and E coincide when

$$0 = c_1 + l_0\sqrt{1-\beta^2} - vt_1,\ t_1 = \frac{c_1 + l_0\sqrt{1-\beta^2}}{v}$$



Thus $\Delta t = t_1 - t_0 = \frac{l_0}{v}\left(1 + \sqrt{1-\beta^2}\right)$

or $\left(\frac{v\,\Delta t}{l_0} - 1\right)^2 = 1-\beta^2 = 1 - \frac{v^2}{c^2}$

From this
$$v = \frac{2c^2\,\Delta t/l_0}{1 + c^2\,\Delta t^2/l_0^2} = \frac{2l_0/\Delta t}{1 + (l_0/c\,\Delta t)^2}$$

1.351 In K_0, the rest frame of the particles, the events corresponding to the decay of the particles are,

$$A: (0,0,0,0) \text{ and } (0, l_0, 0,0) = B$$

In the reference frame K, the corresponding coordintes are by Lorentz transformation

$$A: (0,0,0,0), B: \left(\frac{vl_0}{c^2\sqrt{1-\beta^2}}, \frac{l_0}{\sqrt{1-\beta^2}}, 0,0\right)$$

Now
$$l_0\sqrt{1-\beta^2} = l$$

by Lorentz Fitzgerald contraction formula. Thus the time lag of the decay time of B is

$$\Delta t = \frac{vl_0}{c^2\sqrt{1-\beta^2}} = \frac{v\,l}{c^2(1-\beta^2)} = \frac{vl}{c^2 - v^2}$$

B decays later (B is the forward particle in the direction of motion)

1.352 (a) In the reference frame K with respect to which the rod is moving with velocity v, the coordinates of A and B are

$$A: t, x_A + v(t - t_A), 0, 0$$

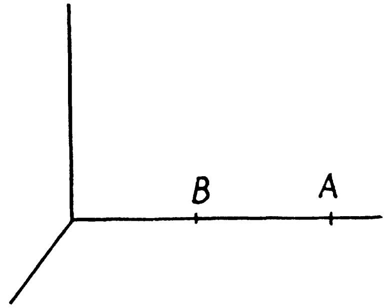
$$B: t, x_B + v(t - t_B), 0, 0$$

Thus $l = x_A - x_B - v(t_A - t_B) = l_0\sqrt{1-\beta^2}$

So $$l_0 = \frac{x_A - x_B - v(t_A - t_B)}{\sqrt{1 - v^2/c^2}}$$

(b) $\pm\, l_0 - v(t_A - t_B) = l = l_0\sqrt{1 - v^2/c^2}$

(since $x_A - x_B$ can be either $+l_0$ or $-l_0$.)



Thus $v(t_A - t_B) = \left(\pm\, 1 - \sqrt{1 - v^2/c_2}\right) l_o$

i.e. $$t_A - t_B = \frac{l_0}{v}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)$$

or $$t_B - t_A = \frac{l_0}{v}\left(1 + \sqrt{1 - v^2/c^2}\right)$$

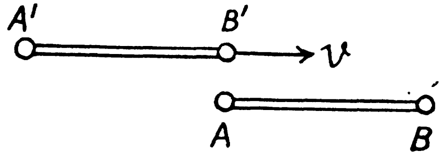
1.353 At the instant the picture is taken the coordintes of A, B, A', B' in the rest frame of $A\,B$ are

$A : (0, 0, 0, 0)$

$B : (0, l_0, 0, 0)$

$B' : (0, 0, 0, 0)$

$A' : (0_1 - l_0\sqrt{1 - v^2/c^2}\,, 0, 0)$



In this frame the coordinates of B' at other times are $B' : (t, vt, 0, 0)$. So B' is opposite to B at time $t(B) = \frac{l_0}{v}$. In the frame in which B', A' is at rest the time corresponding this is by Lorentz tranformation.

$$t^0(B') = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\frac{l_0}{v} - \frac{v l_0}{c^2}\right) = \frac{l_0}{v}\sqrt{1 - v^2/c^2}$$

Similarly in the rest frame of A, B, te coordinates of A at other times are

$$A' : \left(t, -l_0\sqrt{1 - \frac{v^2}{c^2}} + vt, 0, 0\right)$$

A' is opposite to A at time $t(A) = \frac{l_0}{v}\sqrt{1 - \frac{v^2}{c^2}}$

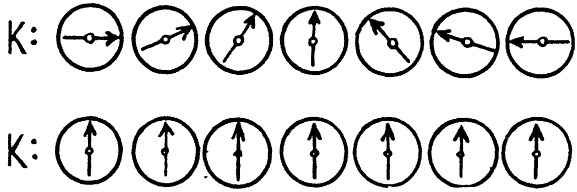
The corresponding time in the frame in which A', B' are at rest is

$$t(A') = \gamma\, t(A) = \frac{l_0}{v}$$

1.354 By Lorentz transfomation $t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)$

So at time $$t = 0,\ t' = \frac{vx}{c^2}\frac{1}{\sqrt{1 - v^2/c^2}}$$

If $x > 0$ $t' < 0$, if $x < 0$, $t' > 0$ and we get the diagram given below "in terms of the K–clock".



The situation in terms of the K' clock is reversed.

1.355 Suppose $x(t)$ is the locus of points in the frame K at which the readings of the clocks of both reference system are permanently identical, then by Lorentz transformation

$$t' = \frac{1}{\sqrt{1 - V^2/c^2}}\left(t - \frac{V\,x(t)}{c^2}\right) = t$$

So differentiating $x(t) = \dfrac{c^2}{V}\left(1 - \sqrt{1 - \dfrac{V^2}{c^2}}\right) = \dfrac{c}{\beta}\left(1 - \sqrt{1 - \beta^2}\right)$, $\beta = \dfrac{V}{c}$

Let $$\beta = \tan h\theta,\ 0 \le \theta < \infty,\ \text{Then}$$

$$x(t) = \frac{c}{\tan h\theta}\left(1 - \sqrt{1 - \tan h^2\theta}\right) = c\,\frac{\cos h\theta}{\sin h\,\theta}\left(1 - \frac{1}{\cos h\,\theta}\right)$$

$$= c\,\frac{\cos h\,\theta - 1}{\sin h\,\theta} = c\sqrt{\frac{\cos h\,\theta - 1}{\cos h\,\theta + 1}} = c\tan h\frac{\theta}{2} \le v$$

·($\tan h\,\theta$ is a monotonically increasing function of θ)

1.356 We can take the coordinates of the two events to be

$$A: (0, 0, 0, 0)\quad B: (\Delta t, a, 0, 0)$$

For B to be the effect and A to be cause we must have $\Delta t > \dfrac{|a|}{c}$.

In the moving frame the coordinates of A and B become

$A: (0, 0, 0, 0), B: \left[\gamma\left(\Delta t - \dfrac{aV}{c^2}\right), \gamma(a - V\Delta t), 0, 0\right]$ where $\gamma = \dfrac{1}{\sqrt{1 - \left(\dfrac{V^2}{c^2}\right)}}$

Since

$$(\Delta t')^2 - \frac{a'^2}{c^2} = \gamma^2\left[\left(\Delta t - \frac{aV}{c^2}\right)^2 - \frac{1}{c^2}(a - V\Delta t)^2\right) = (\Delta t)^2 - \frac{a^2}{c^2} > 0$$

we must have $\Delta t' > \dfrac{|a'|}{c}$

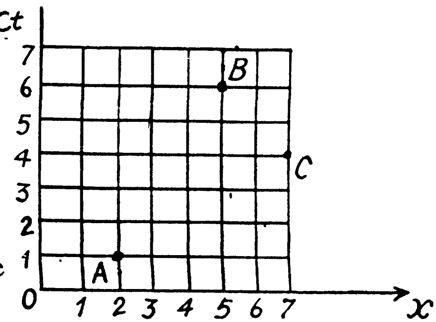
1.357 (a) The four-dimensional interval between A and B (assuming $\Delta y = \Delta z = 0$) is :

$$5^2 - 3^2 = 16 \text{ units}$$

Therefore the time interval between these two events in the reference frame in which the events occurred at the same place is

$$c\,(t'_B - t'_A) = \sqrt{16} = 4\text{ m}$$

or $$t'_B - t'_A = \frac{4}{c} = \frac{4}{3} \times 10^{-8}\text{ s}$$



(b) The four dimensional interval between A and C is (assuming $\Delta y = \Delta z = 0$)

$$3^2 - 5^2 = -16$$

So the distance between the two events in the frame in which they are simultaneous is 4 units = 4m.

1.358 By the velocity addition formula

$$v_x' = \frac{v_x - V}{1 - \dfrac{V v_x}{c^2}}, \quad v'_y = \frac{v_y \sqrt{1 - V^2/c^2}}{1 - \dfrac{v_x V}{c^2}}$$

and $$v' = \sqrt{v'^2_x + v'^2_y} = \frac{\sqrt{(v_x - V)^2 + v_y^2 (1 - V^2/c^2)}}{1 - \dfrac{v_x V}{c^2}}$$

1.359 (a) By definition the velocity of apporach is

$$v_{\text{approach}} = \frac{dx_1}{dt} - \frac{dx_2}{dt} = v_1 - (-v_2) = v_1 + v_2$$

in the reference frame K.

(b) The relative velocity is obtained by the transformation law

$$v_r = \frac{v_1 - (-v_2)}{1 - \dfrac{v_1(-v_2)}{c^2}} = \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$$

1.360 The velocity of one of the rods in the reference frame fixed to the other rod is

$$V = \frac{v + v}{1 + \dfrac{v^2}{c^2}} = \frac{2v}{1 + \beta^2}$$

The length of the moving rod in this frame is

$$l = l_0 \sqrt{1 - \frac{4v^2/c^2}{(1+\beta^2)^2}} = l_0 \frac{1 - \beta^2}{1 + \beta^2}$$

1.361 The approach velocity is defined by

$$\vec{V}_{\text{approach}} = \frac{d\vec{r_1}}{dt} - \frac{d\vec{r_2}}{dt} = V_1 - \vec{V_2}$$

in the laboratory frame. So $V_{\text{approach}} = \sqrt{v_1^2 + v_2^2}$

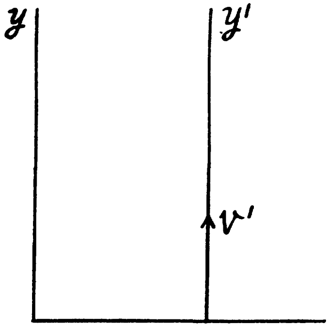
On the other hand, the relative velocity can be obtained by using the velocity addition formula and has the components

$$\left[-v_1,\ v_2\sqrt{1-\left(\frac{v_1^2}{c^2}\right)}\right] \text{ so } V_r = \sqrt{v_1^2 + v_2^2 - \frac{v_1^2 v_2^2}{c^2}}$$

1.362 The components of the velocity of the unstable particle in the frame K are

$$\left(V,\ v'\sqrt{1-\frac{V^2}{c^2}},\ 0\right)$$

so the velocity relative to K is



$$\sqrt{V^2 + v'^2 - \frac{v'^2 V^2}{c^2}}$$

The life time in this frame dilates to

$$\Delta t_0 \Big/ \sqrt{1 - \frac{V^2}{c^2} - \frac{v'^2}{c^2} + \frac{v'^2 V^2}{c^4}}$$

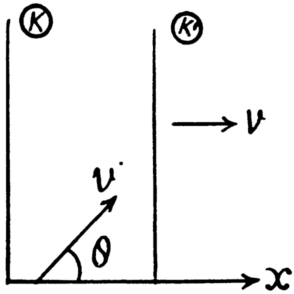
and the distance traversed is

$$\Delta t_0 \frac{\sqrt{V^2 + v'^2 - (v'^2 V^2)/c^2}}{\sqrt{1 - V^2/c^2}\ \sqrt{1 - v'^2/c^2}}$$

1.363 In the frame K' the components of the velocity of the particle are

$$v'_x = \frac{v\cos\theta - V}{1 - \dfrac{vV\cos\theta}{c^2}}$$

$$v'_y = \frac{v\sin\theta\sqrt{1 - V^2/c^2}}{1 - \dfrac{vV}{c^2}\cos\theta}$$



Hence, $\tan\theta' = \dfrac{v'_y}{v'_x} = \dfrac{v\sin\theta}{v\cos\theta - V}\sqrt{(1 - V^2)/c^2}$

1.364 In K' the coordinates of A and B are

$$A : (t', 0, -v't', 0);\ B : (t', l, -v't', 0)$$

After performing Lorentz transformation to the frame K *we get*

$$A : t = \gamma t' \quad B : t = \gamma\left(t' + \frac{Vl}{c^2}\right)$$

$$x = \gamma V t' \quad x = \gamma(l + Vt')$$

$$y = v't' \quad y = -v't'$$

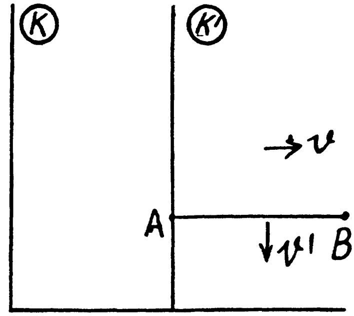
$$z = 0 \quad z = 0$$

By translating $t' \to t' - \dfrac{Vl}{c^2}$, we can write

the coordinates of B as $B : t = \gamma t'$

$$x = \gamma l\left(1 - \frac{V^2}{c^2}\right) + V t' \gamma = l\sqrt{1 - \frac{v^2}{c^2}} + V t' \gamma$$

$$y = -v'\left(t' - \frac{Vl}{c^2}\right), \quad z = 0$$



Thus $$\Delta x = l\sqrt{1 - \left(\frac{V^2}{c^2}\right)}, \quad \Delta y = \frac{v' V l}{c^2}$$

Hence $$\tan\theta' = \frac{v' V}{c^2\sqrt{1 - \frac{v' V}{c^2}}}$$

1.365 t .... $l + dt$ .... Ⓚ
$\vec{v}$ .... $\vec{v} + \vec{w}\,dt$

In K the velocities at time t and $t + dt$ are respectively v and $v + wdt$ along x – axis which is parallel to the vector $\vec{V}$. In the frame K' moving with velocity $\vec{V}$ with respect to K, the velocities are respectively,

$$\frac{v - V}{1 - \frac{vV}{c^2}} \quad \text{and} \quad \frac{v + w\,dt - V}{1 - (v + w\,dt)\frac{V}{c^2}}$$

The latter velocity is written as

$$\frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt}{1 - v\frac{V}{c^2}} + \frac{v - V}{\left(1 - \frac{vV}{c^2}\right)}\frac{w\,V}{c^2}dt = \frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{vV}{c^2}\right)^2}$$

Also by Lorentz transformation

$$dt' = \frac{dt - V\,dx/c^2}{\sqrt{1 - V^2/c^2}} = dt\,\frac{1 - vV/c^2}{\sqrt{1 - V^2/c^2}}$$

Thus the acceleration in the K' frame is

$$w' = \frac{dv'}{dt'} = \frac{w}{\left(1 - \frac{vV}{c^2}\right)^3}\left(1 - \frac{V^2}{c^2}\right)^{3/2}$$

(b) In the K frame the velocities of the particle at the time t and $t + dt$ are repectively

$$(0, v, 0) \text{ and } (0, v + wdt, 0)$$

where $\vec{V}$ is along x–axis. In the K' frame the velocities are

$$\left(-V, v\sqrt{1 - V^2/c^2}, 0\right)$$

and $$\left(-V, (v + wdt)\sqrt{1 - V^2/c^2}, 0\right) \text{ respectively}$$

Thus the acceleration

$$w' = \frac{w dt \sqrt{(1 - V^2/c^2)}}{dt'} = w\left(1 - \frac{V^2}{c^2}\right) \text{ along the } y\text{-axis.}$$

We have used $dt' = \dfrac{dt}{\sqrt{1 - V^2/c^2}}$

1.366 In the instantaneous rest frame $v = V$ and

$$w' = \frac{w}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} \text{ (from 1.365a)}$$

So,

$$= \frac{dv}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} = w'\, dt$$

w' is constant by assumption. Thus integration gives

$$v = \frac{w' t}{\sqrt{1 + \left(\frac{w' t}{c}\right)^2}}$$

Integrating once again $x = \dfrac{c^2}{w'}\left(\sqrt{1 + \left(\dfrac{w' t}{c}\right)^2} - 1\right)$

1.367 The boost time τ_0 in the reference frame fixed to the rocket is related to the time τ elapsed on the earth by

$$\tau_0 = \int_0^\tau \sqrt{1 - \frac{v^2}{c^2}}\, dt = \int_0^\tau \left[1 - \frac{\left(\frac{w' t}{c}\right)^2}{1 + \left(\frac{w' t}{c}\right)^2}\right]^{1/2} dt$$

$$= \int_0^\tau \frac{dt}{\sqrt{1 + \left(\frac{w' t}{c}\right)^2}} = \frac{c}{w'}\int_0^{(w'\tau)/c} \frac{d\xi}{\sqrt{1 + \xi^2}} = \frac{c}{w'} \ln\left[\frac{w'\tau}{c} + \sqrt{1 + \left(\frac{w'\tau}{c}\right)^2}\right]$$

1.368 $m = \dfrac{m_0}{\sqrt{1 - \beta^2}}$

For

$$\beta \approx 1, \quad \frac{m}{m_0} \approx \frac{1}{\sqrt{2(1 - \beta)}} = \frac{1}{\sqrt{2\eta}}$$

1.369 We define the density ρ in the frame K in such a way that $\rho\, dx\, dy\, dz$ is the rest mass dm_0 of the element. That is $\rho\, dx\, dy\, dz = \rho_0\, dx_0\, dy_0\, dz_0$, where ρ_0 is the proper density dx_0, dy_0, dz_0 are the dimensions of the element in the rest frame K_0. Now

$$dy = dy_0, \; dz = dz_0, \; dx = dx_0\sqrt{1 - \frac{v^2}{c^2}}$$

if the frame K is moving with velocity, v relative to the frame K_0. Thus

$$\rho = \frac{\rho_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Defining η by $\rho = \rho_0 (1 + \eta)$

We get $\quad 1 + \eta = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ or, $\dfrac{v^2}{c^2} = 1 - \dfrac{1}{(1+\eta)^2} = \dfrac{\eta\,(2+\eta)}{(1+\eta)^2}$

or
$$v = c\sqrt{\frac{\eta\,(2+\eta)}{(1+\eta)^2}} = \frac{c\sqrt{\eta\,(2+\eta)}}{1+\eta}$$

1.370 We have

$$\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = p \quad \text{or,} \quad \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{m_0^2 + \frac{p^2}{c^2}}$$

or
$$1 - \frac{v^2}{c^2} = \frac{m_0^2 c^2}{m_0^2 c^2 + p^2} = 1 - \frac{p^2}{p^2 + m_0^2 c^2}$$

or
$$v = \frac{c_p}{\sqrt{p^2 + m_0^2 c^2}} = \frac{c}{\sqrt{1 + \left(\frac{m_0 c}{p}\right)^2}}$$

So $\quad \dfrac{c - v}{c} = \left[1 - \left(1 + \left(\dfrac{m_0 c}{p}\right)^2\right)^{-1/2}\right] \times 100\ \% = \dfrac{1}{2}\left(\dfrac{m_0 c}{p}\right)^2 \times 100\ \%$

1.371 By definition of η,

$$\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} = \eta\, m_0 v \quad \text{or} \quad 1 - \frac{v^2}{c^2} = \frac{1}{\eta^2}$$

or
$$v = c\sqrt{1 - \frac{1}{\eta^2}} = \frac{c}{\eta}\sqrt{\eta^2 - 1}$$

1.372 The work done is equal to change in kinetic energy which is different in the two cases Classically i.e. in nonrelativistic mechanics, the change in kinetic energy is

$$\frac{1}{2} m_0 c^2 \left((0{\cdot}8)^2 - (0{\cdot}6)^2\right) = \frac{1}{2} m_0 c^2\, 0{\cdot}28 = 0{\cdot}14\, m_0 c^2$$

Relativistically it is,

$$\frac{m_0 c^2}{\sqrt{1 - (0{\cdot}8)^2}} - \frac{m_0 c^2}{\sqrt{1 - (0{\cdot}6)^2}} = \frac{m_0 c^2}{0{\cdot}6} - \frac{m_0 c^2}{0{\cdot}8} = m_0 c^2 (1{\cdot}666 - 1{\cdot}250)$$

$$= 0{\cdot}416\, m_0 c^2 = 0{\cdot}42\, m_0 c^2$$

1.373 $$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2 m_0 c^2$$

or $$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \quad \text{or} \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

or $$\frac{v}{c} = \frac{\sqrt{3}}{2} \cdot \text{i.e.} \quad v = c\frac{\sqrt{3}}{2}$$

1.374 Relativistically

$$\frac{T}{m_0 c^2} = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4$$

So $$\beta_{rel}^2 \approx \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\beta_{rel}^2\right)^2 \approx \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\frac{2T}{m_0 c^2}\right)^2$$

Thus $$\beta_{rel} = \left[\frac{2T}{m_0 c^2} - 3\frac{T^2}{m_0^2 c^4}\right]^{1/2} = \sqrt{\frac{2T}{m_0 c^2}}\left(1 - \frac{3}{4}\frac{T}{m_0 c^2}\right)$$

But Classically, $\beta_{cl} = \sqrt{\frac{2T}{m_0 c^2}}$ so $\frac{\beta_{rel} - \beta_{cl}}{\beta_{cl}} = \frac{3}{4}\frac{T}{m_0 c^2} = \varepsilon$

Hence if $$\frac{T}{m_0 c^2} < \frac{4}{3}\varepsilon$$

the velocity β is given by the classical formula with an error less than ε.

1.375 From the formula

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad p = \frac{m_0 v}{\sqrt{1-v^2/c^2}}$$

we find $$E^2 = c^2 p^2 + m_0^2 c^4 \quad \text{or} \quad (m_0 c^2 + T)^2 = c^2 p^2 + m_0^2 c^4$$

or $$T(2 m_0 c^2 + T) = c^2 p^2 \quad \text{i.e.} \quad p = \frac{1}{c}\sqrt{T(2m_0 c^2 + T)}$$

1.376 Let the total force exerted by the beam on the target surface be F and the power liberated there be P. Then, using the result of the previous problem we see

$$F = Np = \frac{N}{c}\sqrt{T(T + 2m_0 c^2)} = \frac{I}{ec}\sqrt{T(T + 2m_0 c^2)}$$

since $I = Ne$, N being the number of particles striking the target per second. Also,

$$P = N\left(\frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2\right) = \frac{I}{e}T$$

These will be, respectively, equal to the pressure and power developed per unit area of the target if I is current density.

1.377 In the frame fixed to the sphere :- The momentum transferred to the eastically scatterred particle is

$$\frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

The density of the moving element is, from 1.369, $n\dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

and the momentum transferred per unit time per unit area is

$$p = \text{ the pressure} = \frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}\, n \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot v = \frac{2\,mn\,v^2}{1-\frac{v^2}{c^2}}$$

In the frame fixed to the gas :- When the sphere hits a stationary particle, the latter recoils with a velocity

$$= \frac{v+v}{1+\frac{v^2}{c^2}} = \frac{2\,v}{1+\frac{v^2}{c^2}}$$

The momentum transferred is $\dfrac{\dfrac{m\cdot 2v}{1+v^2/c^2}}{\sqrt{1-\dfrac{4v^2/c^2}{(1-v^2/c^2)^2}}} = \dfrac{2\,mv}{1-\frac{v^2}{c^2}}$

and the pressure is $\dfrac{2\,mv}{1-\frac{v^2}{c^2}} \cdot n \cdot v = \dfrac{2\,mnv^2}{1-\frac{v^2}{c^2}}$

1.378 The equation of motion is

$$\frac{d}{dt}\left(\frac{m_0\,v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F$$

Integrating $= \dfrac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{\beta}{\sqrt{1-\beta^2}} = \dfrac{F_t}{m_0\,c}$, using $v = 0$ for $t = 0$

$$\frac{\beta^2}{1-\beta^2} = \left(\frac{Ft}{m_0\,c}\right)^2 \quad \text{or,} \quad \beta^2 = \frac{(Ft)^2}{(Ft)^2+(m_0\,c)^2} \quad \text{or,} \quad v = \frac{Fct}{\sqrt{(m_0\,c)^2+(Ft)^2}}$$

$$\text{or} \quad x = \int \frac{Fct\;dt}{\sqrt{F^2t^2+m_0^2c^2}} = \frac{c}{F}\int \frac{\xi\,d\xi}{\sqrt{\xi^2+(m_0\,c)^2}} = \frac{c}{F}\sqrt{F^2t^2+m_0^2c^2} + \text{constant}$$

or using $x = 0$ at $t = 0$, we get, $x = \sqrt{c^2t^2+\left(\dfrac{m_0\,c^2}{F}\right)^2} - \dfrac{m_0\,c^2}{F}$

1.379 $x = \sqrt{a^2 + c^2 t^2}$, so $\dot{x} = v = \dfrac{c^2 t}{a^2 + c^2 t^2}$

or, $$\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c^2 t}{a}. \text{ Thus } \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{m_0 c^2}{a} = F$$

1.380 $$\vec{F} = \frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = m_0 \frac{\dot{\vec{v}}}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 \frac{\vec{v}}{c^2} \vec{v} \cdot \dot{\vec{v}} \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

Thus $$\vec{F}_{\perp} = m_0 \frac{\vec{w}}{\sqrt{1 - \beta^2}}, \quad \vec{w} = \dot{\vec{v}}, \quad \vec{w}_{\perp} \vec{v}$$

$$\vec{F}_{\parallel} = m_0 \frac{\vec{w}}{(1 - \beta^2)^{3/2}}, \quad \vec{w} = \dot{\vec{v}}, \quad \vec{w}_{\parallel} \vec{v}$$

1.381 By definition,

$$E = m_0 \frac{c^2}{\sqrt{1 - \frac{v_x^2}{c^2}}} = \frac{m_0 c^3 dt}{ds}, \quad p_x = m_0 \frac{v_x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c\, m_0\, dx}{ds}$$

where $ds^2 = c^2 dt^2 - dx^2$ is the invariant interval ($dy = dz = 0$)

Thus, $$p'_x = cm_0 \frac{dx'}{ds} = cm_0 \gamma \frac{(dx - Vdt)}{ds} = \frac{p_x - VE/c^2}{\sqrt{1 - V^2/c^2}}$$

$$E' = m_0 c^3 \frac{dt'}{ds} - c^3 m_0 \gamma \frac{\left(dt - \frac{Vd_x}{c^2}\right)}{ds} = \frac{E - Vp_x}{\sqrt{1 - \frac{V^2}{c^2}}},$$

1.382 For a photon moving in the x direction

$$\varepsilon = cp_x, \; p_y = p_z = 0,$$

In the moving frame, $\varepsilon' = \dfrac{1}{\sqrt{1 - \beta^2}}\left(\varepsilon - V \dfrac{\varepsilon}{c}\right) = \varepsilon \sqrt{\dfrac{1 - V/c}{1 + V/c}}$

Note that $\varepsilon' = \dfrac{\varepsilon}{2}$ if, $\dfrac{1}{4} = \dfrac{1 - \beta}{1 + \beta}$ or $\beta = \dfrac{3}{5}$, $V = \dfrac{3c}{5}$.

1.383 As before

$$E = m_0 c^3 \frac{dt}{ds}, \; p_x = m_0 c \frac{dx}{ds}.$$

Similarly $$p_y = m_0 c \frac{dy}{ds}, \; p_z = m_0 c \frac{dz}{ds}$$

Then $$E^2 - c^2 p^2 = E^2 - c^2 \left(p_x^2 + p_y^2 + p_x^2\right)$$

$$= m_0^2 c^4 \frac{(c^2 dt^2 - dx^2 - dy^2 - dz^2)}{ds^2} = m_0^2 c^4 \text{ is invariant}$$

1.384 (b) & (a) In the CM frame, the total momentum is zero, Thus

$$\frac{V}{c} = \frac{cp_{1x}}{E_1 + E_2} = \frac{\sqrt{T(T + 2m_0 c^2)}}{T + 2m_0 c^2} = \sqrt{\frac{T}{T + 2m_0 c^2}}$$

where wc have used the result of probiem (1.375)

Then

$$\frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - \dfrac{T}{T + 2m_0 c^2}}} = \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}}$$

Total energy in the CM frame is

$$\frac{2m_0 c^2}{\sqrt{1 - V^2/c^2}} = 2m_0 c^2 \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}} = \sqrt{2 m_0 c^2 (T + 2m_0 c^2)} = \tilde{T} + 2m_0 c^2$$

So $$\tilde{T} = 2m_0 c^2 \left(\sqrt{1 + \frac{T}{2m_0 c^2}} - 1\right)$$

Also $2\sqrt{c^2 \tilde{p}^2 + m_0^2 c^4} = \sqrt{2m_0 c^2 (T + 2m_0 c^2)}$, $4 c^2 \tilde{p}^2 = 2m_0 c^2 T$, or $\tilde{p} = \sqrt{\frac{1}{2} m_0 T}$

1.385 $M_0 c^2 = \sqrt{E^2 - c^2 p^2}$

$$\sqrt{(2m_0 c^2 + T)^2 - T(2m_0 c^2 + T)} = \sqrt{2m_0 c^2 (2m_0 c^2 + T)} = c\sqrt{2m_0 (2m_0 c^2 + T)}$$

Also $$cp = \sqrt{T(T + 2m_0 c^2)}, \; v = \frac{c^2 p}{E} = c\sqrt{\frac{T}{T + 2m_0 c^2}}$$

1.386 Let T' = kinetic energy of a proton striking another stationary particle of the same rest mass. Then, combined kinetic energy in the CM frame

$$= 2m_0 c^2 \left(\sqrt{1 + \frac{T'}{2m_0 c^2}} - 1\right) = 2T, \; \left(\frac{T}{m_0 c^2} + 1\right)^2 = 1 + \frac{T'}{2m_0 c^2}$$

$$\frac{T'}{2m_0 c^2} = \frac{T(2m_0 c^2 + T)}{m_0^2 c^4}, \; T' = \frac{2T(T + 2m_o c^2)}{m_0 c^2}$$

1.387 We have

$$E_1 + E_2 + E_3 = m_0 c^2, \ \vec{p_1} + \vec{p_2} + \vec{p_3} = 0$$

Hence $$(m_0 c^2 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (E_2 + E_3)^2 - \left(\vec{p_2} + \vec{p_3}\right)^2 c^2$$

The L.H.S. $= (m_0^2 c^4 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (m_0^2 + m_1^2) c^4 - 2m_0 c^2 E_1$

The R.H.S. is an invariant. We can evaluate it in any frame. Choose the CM frame of the particles 2 and 3.

In this frame R.H.S. $= (E'_2 + E'_3)^2 = (m_2 + m_3)^2 c^4$

Thus $(m_0^2 + m_1^2) c^4 - 2m_0 c^2 E_1 = (m_2 + m_3)^2 c^4$

or $2 m_0 c^2 E_1 \le \left\{ m_0^2 + m_1^2 - (m_2 + m_3)^2 \right\} c^4$, or $E_1 \le \dfrac{m_0^2 + m_1^2 - (m_2 + m_3)^2}{2 m_0} c^2$

1.388 The velocity of ejected gases is u realtive to the rocket. In an earth centred frame it is

$$\frac{v - u}{1 - \dfrac{vu}{c^2}}$$

in the direction of the rocket. The momentum conservation equation then reads

$$(m + dm)(v + dv) + \frac{v - u}{1 - \dfrac{uv}{c^2}} (- dm) = mv$$

or $$mdv - \left(\frac{v - u}{1 - \dfrac{uv}{c^2}} - v \right) dm = 0$$

Here $-dm$ is the mass of the ejected gases. so

$$mdv - \frac{-u + \dfrac{uv^2}{c^2}}{1 - \dfrac{uv}{c^2}} dm = 0, \text{ or } mdv + u\left(1 - \frac{v^2}{c^2}\right) dm = 0$$

(neglecting $1 - \dfrac{uv}{c^2}$ since u is non- relativistic.)

Integrating $\left(\beta = \dfrac{v}{c}\right)$, $\displaystyle\int \frac{d\beta}{1 - \beta^2} + \frac{u}{c}\int \frac{dm}{m} = 0$, $\ln \dfrac{1 + \beta}{1 - \beta} + \dfrac{u}{c} \ln m = \text{constant}$

The constant $= \dfrac{u}{c} \ln m_0$ since $\beta = 0$ initially.

Thus $$\frac{1 - \beta}{1 + \beta} = \left(\frac{m}{m_0}\right)^{u/c} \quad \text{or} \quad \beta = \frac{1 - \left(\dfrac{m}{m_0}\right)^{u/c}}{1 + \left(\dfrac{m}{m_0}\right)^{u/c}}$$